These may be chosen as follows, both surfaces being of the same material:

<table>
<thead>
<tr>
<th>Material</th>
<th>Force (pounds)</th>
<th>Diameter (inches)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cast Iron</td>
<td>14,425,000</td>
<td>1000 to 1400</td>
</tr>
<tr>
<td>Wrought Iron</td>
<td>28,440,000</td>
<td>1000 to 1400</td>
</tr>
<tr>
<td>Steel (hardened)</td>
<td>43,660,000</td>
<td>1000 to 1400</td>
</tr>
</tbody>
</table>

Example 1. The bridge over the Rhine at Boppard has spans of 330 feet. The beams are made of cast iron of the form shown at A. The propense is 771,000 pounds on six rollers, the dimensions of the latter being, \( r = 30 \), \( v = 4.153 \).

We have therefore \( F = \frac{P}{r} = \frac{771,000}{30} = 25,700 \), hence \( \beta = \frac{2}{\sqrt{\frac{41}{4}} \sqrt{\frac{69.8}{0.156}}} = 0.035 \).

This gives the breadth \( b \) of the contact surface under this load, \( b = \beta r = \frac{4.153 \times 0.035}{0.27}, \text{and} \quad S = \frac{15}{16} \times 771,000 = 12,915 \text{ lb} \).

Example 2. Bridge over the Rhine at Wiesel: span 425 feet rollers and bearings of hardened steel. The load is 700,000 pounds on six rollers, as shown at C, and \( r = 37.35 \), \( v = 3.653 \). These values give \( \beta = 0.415 \) and \( \alpha = 0.43 \), and \( S = 34.90 \text{ lb} \).

Example 3. Clifton Bridge at Niagara. The load of 771,000 lb is carried upon 11 steel rollers, on bearings of the same material, their dimensions being \( r = 63 \), \( v = 66.9 \). This gives a high value for \( \beta = 4.172, \beta = 0.37, \) hence \( b = 0.402, \text{and} \quad S = 74.25 \text{ lb} \).

Ball bearings are frequently used instead of cylindrical bearings, and for some forms of journals are most convenient, although the bearing surfaces being only points, they are not so well adapted for heavy pressure.

The above described apparatus is fitted to both ends of the rollers. In order to provide for any slight inequality in diameter between the opposite ends of the rollers, another adjustment of the rollers, another adjustment is provided. This consists of the lever \( a \), to which the planet roller is suspended by the link \( a \). This permits the planet roller to be forced into the narrower space between \( r \) and \( R \), by means of the worm and worm sector shown at \( e \). The ring \( R \) is a continuous steel forging, and the rollers are chilled castings. The rollers \( R \), \( R \), are geared together, the gears having double spiral teeth, as shown in Fig. 569.

Fiction rollers are sometimes used in connection with toothed gearing, an example of which will be seen later in Fig. 569. Whitworth used rollers in the place of a nut in his screw planing machine, and they are used in worm gearing by Bourdon, and the higher form of worm, the globoid (see § 224) by Jensen, and by Hawkins. Many applications are also found among instruments of precision, notably Atwood's Machine, and Andler's Planimeter.

A form of roller bearing used in agricultural machinery is that of Cambron, shown in Fig. 567. The steel ring, with semicircular groove is secured to the shaft, and in the groove, or globoid ring, the steel balls, 93 to 13 in number, are placed. These are held in place by a corresponding external ring, made in halves. The outer ring is held in the journal box. Cambron uses balls of \( \frac{1}{4} \) in diameter, which are rolled in a mill of similar construction to the bearing.

It may be remarked that when roller bearings are used for car or wagon wheels, we have a combination of friction wheels, since the wheels themselves are properly friction rollers and the introduction of rollers into the axle bearing makes the latter device what may be termed friction wheels of the second order.

Another system of the higher order is the very ingenious arrangement of planet rollers of Meckwitz. In this apparatus, which seems to have been so completely conceived by the inventor as to be incapable of further improvement, friction rollers are utilized to the fullest possible extent. The system is illustrated in the diagram Fig. 568. \( R_1, R_2, R_3 \), are the rollers of a roller-mill. The axis \( a \) of the roller \( R_1 \), is carried in a stationary bearing, the axes \( b \) and \( c \) are carried from links \( a, b, \text{and} \ a_c \), so that they may be moved to or from \( a \), or may be pressed against the latter. The length of these links is governed by a screw adjustment. The rollers are pressed together by the ring roller \( K \), acting upon the planet roller \( r \), and rollers \( s \), \( t \), the latter being loose upon the axes of the main rollers \( K_1 \) and \( K_2 \). The planet roller \( r \) acts against the roller \( r \), which is fast on the axis of \( K_3 \). If it is desired to exert greater pressure upon the rollers, the roller \( K_3 \) is forced towards \( K_2 \), by means of the lever combination \( a, b, c, d, \text{and} \ a_c \), the lever \( d, c \) being held in position by a ratchet section, the position being changed as the rollers wear. The ring roller \( K_3 \) reduces very greatly the wear upon the journals of the grinding rollers, as it converts the greater part of their journal friction into rolling friction. In order to equalize the effect of the weight of the upper roller, the lower roller \( K_1 \) is counterbalanced by a weight, which, acting through the system of levers \( c, d, a, \text{and} \ c_a \), exerts an upward pressure equal to the combined weight of \( R_1 \) and \( R_2 \).
base form is the hyperboloid, which name is also given to the gears. For many applications of inclined axes the teeth are made spiral, giving the various forms of spiral gears and worm gears.

If the motion is to be transmitted at a uniform rate, the base figures are solids of revolution (cylinders, cones, hyperboloids); the wheels themselves being round, while if the motion is not to be transmitted uniformly the outlines will be irregular. In the following discussion only round gear wheels will be considered.

A. THE CONSTRUCTION OF SPUR TEETH.

§ 200.

GENERAL CONSIDERATIONS.

The form of gear teeth may be so chosen that all gears of the same pitch will work together. Wheels of this sort are called interchangeable, while wheels which are not so made will run only in pairs.

In each pair of round wheels there are two circles, struck from the centres of the wheels, which have at each moment the same linear velocity, and are called in general the ratio circles. The particular ratio circles for a pair of spur gear wheels are called their pitch circles. Upon these circumscribing circles are laid out the pitch divisions, i.e., the space from centre to centre of the teeth.

The teeth themselves are prismatic in shape, the base of the prism being the outline of the tooth. The portion of the tooth which projects beyond the pitch cylinder is called the point of the tooth, and that portion within the pitch cylinder is the base.

The surface of the base is called the faces of the tooth, and the surfaces of the base, the flanks.

Fig. 569.

In spur gear teeth we also have, Fig. 569, the length \( l \), the breadth on face \( b \), the tooth thickness \( t \), of the pitch being indicated by \( p \), the two latter being measured on the curve of the pitch circle.

The teeth in one and the same wheel are made of the same thickness and same spacing, so that any tooth will fit into any space. It follows from this that, the spaces being made of suitable size to receive the teeth, that the inverse ratio of the number of revolutions \( n \) and \( n_1 \) of a pair of wheels is equal to the direct ratio of the respective numbers of teeth \( Z \) and \( Z_1 \), or:

\[
\frac{n_1}{n} = \frac{Z}{Z_1} \tag{186}
\]

This statement is equally true for circular and non-circular wheels. It also holds good if the thickness of the teeth is different at different portions of the circumference, providing only that care is taken that the spaces in the smaller gear come around to match their proper teeth each revolution. If, therefore, under these conditions we have the number of teeth given for any case, we may consider the above relation as the fundamental formula of the transmission of motion by toothed gearing. This is rather to be considered as an inevitable principle of construction rather than a fine geometrical distinction. It depends upon the primitive form of gear construction which has been in use for centuries in the Orient, where no other care is taken in the proportioning of gears except that they are large enough and that the pin teeth are sufficiently strong.

No general principle can be laid down for the form of the flanks of teeth. For round wheels, the ratio of the angular velocities, \( \dot{\alpha} \), of that of the differentials of the simultaneous angles of rotation \( \alpha_1 \) and \( \alpha \) must equal the ratio \( \frac{Z_2}{Z_1} \). This affects the flanks as being those surfaces upon which the ratio \( \frac{d\alpha_1}{d\alpha} \) depends.

The form of the teeth is of great importance. Especially necessary is it that the division of the pitch shall be teeth are made exactly; errors in shape of the flanks are even less injurious than errors in the dividing. Accurate spacing can only be accomplished by the use of suitable gear-cutting machinery, and such machines are now in general use.

Accuracy in spacing is of especial importance in the change gears of a lathe, as any error would correspond to a defect in the screw which is being cut. Such defects are still more apparent if the lathe is used for cutting spiral gears (see § 227, below). The smooth motion which must; spiral gears are intended to produce may thus be prevented by irregular cutting. The choice of tooth outline to be adopted, either for the entire product of an establishment; or for any class of work, can only be made after a careful consideration of all the conditions upon which so much depends. These considerations will be taken up and discussed in the following sections:

§ 201.

PITCH RADIUS. CIRCUMFERENTIAL DIVISION.

For any pitch \( p \) and number of teeth \( Z \) for a round wheel we have for the radius \( K \) of the pitch circle:

\[
\frac{R}{i} = \frac{Z}{2} = \frac{0.1953 Z}{2} \tag{187}
\]

which gives, according to formula (186)

\[
\frac{K}{i} = \frac{Z}{2} \tag{188}
\]

The radius obtained from formula (187) is never a whole number, because \( \pi \) is an irrational number, so that \( K \) will always contain a fraction if the pitch is a whole number. The following table will facilitate the computation in such cases. If the irrational feature is to be kept out of the value of \( K \), the length of the pitch divisions must not be made with any instruments, but formed by multiples of \( \pi \) or of \( \pi / Z \) and such multiples is used to mark the teeth.

If we call the pitch \( p \), we have under this plan:

\[
K = \frac{Z}{2} \left( \frac{\pi}{1} \right) \tag{189}
\]

This corresponds to the so-called "diametrical pitch" system of England and America.

Example: Suppose a wheel of 36 teeth and a pitch of \( \pi \times 3.1416 \) mm, we have according to (187), for the radius \( K \) of its pitch circle, \( K = \frac{Z}{2} \times 0.1953 \times 6.74 \) mm, and if we have in English units a pitch of \( \pi \times 3.1416 \) for a wheel of 36 teeth, we have according to (187), \( K = \frac{Z}{2} \times 3 = \frac{18}{2} \times 3 = 15 \) in.

A convenient instrument in this connection is a circumference scale. This consists of a prismatic rule of wood or metal upon which, for the metric system, a length of \( 31.416 \) millimetres is laid out, and on a parallel line the same distance is divided into 100 equal parts. Corresponding points on the two scales will then have to each other the ratio \( 1 : \pi \). This scale is useful for the rectification of circles and circular arcs. Similar scales may be prepared upon sixteenths, tenths or any subdivision of the inch.

In the following discussion both methods will be used, namely: that in which the pitch is taken in rational numbers, thus making the radius irrational; and that in which the pitch is made rational in units of the circumference scale, and hence the radius becomes rational. The following table is not to be confused with that of Doukind, * made according to the expression,

\[
\frac{p}{i} = \frac{Z}{2} \left( \frac{\pi}{1} \right) \tag{187}
\]

which gives the radius of the circumscribing circle of a regular polygon of \( Z \) sides, each having a length equal to \( \frac{p}{i} \). This latter radius differs from the radius \( K \) above referred to for small values of \( Z \), and confusion in this respect has given rise to numerous errors.

§ 202.

TABLE OF RADIUS OF PITCH CIRCLES.

Examples in the use of the following table. (Note. This table was calculated for use with the metric system, in which the pitch is generally taken in millimetres. It may, however, be used equally well in English units, by taking the pitch in sixteenths, in order to make the divisions sufficiently small.)

Example 1. A wheel of 64 teeth, and \( \pi / 2 \) pitch is to be made required the radius of the pitch circle. The pitch circle be taken 36 arc minutes, and we have at the intersection of the columns for 64 and \( \pi / 2 \), the number 3215, hence \( K = 17.954 \text{ in.} \). This gives a pitch of \( \pi / 2 \) giving a diameter of \( 33 \text{ in.} \).

The table may also be used to determine the number of teeth when the pitch is chosen and the radius given.

*See Selsenger’s Vorlage, p. 93, and others.
Example 2. Given a wheel of 50 inches radius and 16° pitch. This gives $R = 20.03$. The nearest value to this in the table is 20.00 at the intersection of 50, and 7, and hence 157 is the radius of the pitch circle.

When the radius and number of teeth are given the table may be used to find the pitch.

Example 3. Given $R = 25°$, $x = 16$. We find in the table at the intersection of 20 and 4, the value of $R = 6.40$. Then we have $P = 6.40$. The value of $P$ is 6.40.

<table>
<thead>
<tr>
<th>Z</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.00</td>
<td>0.15</td>
<td>0.31</td>
<td>0.47</td>
<td>0.63</td>
<td>0.79</td>
<td>0.95</td>
<td>1.11</td>
<td>1.28</td>
<td>1.44</td>
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<tr>
<td>1</td>
<td>1.02</td>
<td>1.37</td>
<td>1.70</td>
<td>2.03</td>
<td>2.35</td>
<td>2.67</td>
<td>3.00</td>
<td>3.33</td>
<td>3.65</td>
<td>3.98</td>
</tr>
<tr>
<td>2</td>
<td>1.50</td>
<td>1.90</td>
<td>2.30</td>
<td>2.69</td>
<td>3.07</td>
<td>3.44</td>
<td>3.81</td>
<td>4.17</td>
<td>4.52</td>
<td>4.87</td>
</tr>
<tr>
<td>3</td>
<td>1.67</td>
<td>2.16</td>
<td>2.63</td>
<td>3.09</td>
<td>3.53</td>
<td>3.96</td>
<td>4.38</td>
<td>4.80</td>
<td>5.20</td>
<td>5.59</td>
</tr>
<tr>
<td>4</td>
<td>1.57</td>
<td>2.08</td>
<td>2.58</td>
<td>3.06</td>
<td>3.52</td>
<td>3.98</td>
<td>4.44</td>
<td>4.89</td>
<td>5.33</td>
<td>5.76</td>
</tr>
<tr>
<td>5</td>
<td>2.44</td>
<td>3.00</td>
<td>3.56</td>
<td>4.11</td>
<td>4.64</td>
<td>5.16</td>
<td>5.67</td>
<td>6.17</td>
<td>6.66</td>
<td>7.14</td>
</tr>
<tr>
<td>6</td>
<td>3.47</td>
<td>4.13</td>
<td>4.77</td>
<td>5.40</td>
<td>5.99</td>
<td>6.56</td>
<td>7.11</td>
<td>7.65</td>
<td>8.18</td>
<td>8.70</td>
</tr>
<tr>
<td>7</td>
<td>5.06</td>
<td>5.74</td>
<td>6.40</td>
<td>7.04</td>
<td>7.66</td>
<td>8.27</td>
<td>8.87</td>
<td>9.46</td>
<td>10.04</td>
<td>10.62</td>
</tr>
<tr>
<td>8</td>
<td>10.22</td>
<td>11.06</td>
<td>11.89</td>
<td>12.70</td>
<td>13.49</td>
<td>14.27</td>
<td>15.03</td>
<td>15.79</td>
<td>16.54</td>
<td>17.28</td>
</tr>
<tr>
<td>9</td>
<td>11.54</td>
<td>12.36</td>
<td>13.17</td>
<td>13.97</td>
<td>14.76</td>
<td>15.54</td>
<td>16.32</td>
<td>17.09</td>
<td>17.86</td>
<td>18.62</td>
</tr>
</tbody>
</table>

The General Solution of Tooth Outlines.

Two tooth outlines which work together lie in a section at right angles to the axes of the wheels and in general this section the construction and action of the teeth is to be considered. The so-called general solution of tooth outlines is that by which, if a form of tooth be given for one wheel, the proper form of the other may be drawn so that the motion will be transmitted with a uniform velocity ratio. Several such solutions will be given.

I. The Author's First Solution. Given the tooth

The Action of Tooth Outlines.

In order to find the necessary strength it is frequently desirable to make the root of the tooth as thick as can be done without interfering with the path of the face of the corresponding tooth of the other gear. This path will be determined in the following manner. Let $a$, $b$, $c$, $d$, $e$, etc., be the points of contact for the wheel $T$, $S$, $T'$, etc., that the tangent $S'N$ of the flank outline for the latter tooth, and $1$ $S$ is the action line between the limits of the outside diameter circles $S$ and $K$. Lay off from $S$ on both pitch circles the corresponding spaces $S1$, $S2$, $S3$, etc., and the envelope of these will give the path of the flank outline. The actual profile of the flank is drawn tangent to the corresponding tooth outlines to the point where it crosses the clearance circle $T'$. The theoretical curve is a prolongation of the profile curve (see § 206). In the figure in which $T$ is a straight line, or rack, the curve is an abridged evolute.

II. Abridged Solution. (Pendel). Fig 571. Mark off on the pitch circle $T'$, the points $a$, $b$, $c$, $d$, etc., which roll into contact with points $a$, $b$, $c$, etc., of the work $T$, draw from $a$, $b$, $c$, etc., arcs with radii respectively equal in length to the normals to the given tooth outline $a$, $b$, etc., then will a curve drawn tangent to these arcs be the required outline. This method, with the corresponding points of $T$, $S$, etc., instead of forward, the outline for an internal gear will be obtained for the wheel $O'$.

III. The Author's Second Solution. Fig. 572. The tooth outline $a$, $b$, $c$, $d$, $e$, etc., is given, and its pitch circle $T'$, also the pitch circle $T$. Draw the normals $a$, $b$, $c$, etc., also draw from $O$, as a centre, arcs through $a$, $b$, $e$, etc., and make $S1 = a$, $SII = b$, $SIII = c$, etc., and draw the curve I, II, III, S, IV, V, etc.; this curve will be the path of the point of contact of the tooth, and may be called the Line of Action.

IV. Theoretical Profile of the Flank. Fig. 573. In order to obtain the necessary strength it is frequently desirable to make the root of the tooth as thick as can be done without interfering with the path of the face of the corresponding tooth of the other gear. This path will be determined in the following manner. Let $a$, $b$, $c$, $d$, etc., be the points of contact for the wheel $T$, $S$, $T'$, etc., that the tangent $S'N$ of the flank outline for the latter tooth, and $1$ $S$ the line of action between the limits of the outside diameter circles $S$ and $K$. Lay off from $S$ on both pitch circles the corresponding spaces $S1$, $S2$, $S3$, etc., and the envelope of these will give the path of the flank outline. The actual profile of the flank $a$, $b$, $c$, etc., is drawn tangent to the corresponding tooth outlines to the point where it crosses the clearance circle $T'$. The theoretical curve is a prolongation of the profile curve (see § 206). In the figure in which $T$ is a straight line, or rack, the curve is an abridged evolute.

First discussed in Moll & kneitscher's "Konstruktionslehre für den Maschinenbau."
action and the initial point of contact is called the rolling arc for the given point. For example $S + S$ is the rolling arc on $T$, for the point $T$, and $S + S$ on $T$, for the same point.

The sum of the rolling arcs between the two extreme points (arc 1 $S + S$, or arc 1 $S + S$) is the arc of action, and its length indicates the duration of the action of the given pair of teeth, which is easily determined graphically. It depends upon the length of that portion of the line of action which it is desired to use. This is usually taken between the limits of the circles of the outside and the case of the teeth, which gives in Fig. 572 the line of action VI.

For any wheel of given tooth outline and pitch diameter there is but one line of action, and for a given line of action but one tooth profile. This latter can only be determined from the line of action when the rolling arc for the pitch point of the line of action are also given.

For cycloidal teeth the rolling arc is also the line of action and for this reason the geometrical discussion is much simplified. In order that a pair of gear wheels should work properly together, their lines of action should correspond and their rolling arcs be of equal length for homologous points of action. By conforming to these conditions any number of gear wheels may be made to operate with a given wheel. Such wheels are said to be interchangeable or series wheels, since the common line of action is symmetrically disposed on each side of the pitch circle, as well as on each side of a radial line passing through its pitch point.

The ray drawn from the pitch point through any point on the line of action (as $N$, in Fig. 572) gives the direction of the pressure between the teeth for that point.

**Fig. 574.**

### THE CYCLOIDAL CURVES.

For the generation of tooth outlines for gears to be used interchangeably in series, the cycloidal curves or those produced by rolling circles are the best. When one circle rolls upon another in the same plane without sliding, each point in any radius describes a curve which is called either a common, extended or abridged cycloid, according as the point is situated on the circumference of the circle, or on a radial line without or within the circumference.

The stationary circle is the base circle of the curve, and its radius will be here indicated by $R$; and the radius of the rolling circle by $r$. If we consider either radius negative when it lies within the other circle, and positive when it lies without, we may distinguish the five kinds of cycloidal curves whose radii have the relation $R$, and $r$, in the following manner:

<table>
<thead>
<tr>
<th>Base Circle</th>
<th>Rolling Circle</th>
<th>Corresponding Curve Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$+R$</td>
<td>$+r$</td>
<td>Epicycloid</td>
</tr>
<tr>
<td>$+R$</td>
<td>$-r$</td>
<td>Orthocycloid</td>
</tr>
<tr>
<td>$-R$</td>
<td>$+r$</td>
<td>Hypocycloid</td>
</tr>
<tr>
<td>$+R$</td>
<td>$\pm\infty$</td>
<td>Epitrochoid</td>
</tr>
<tr>
<td>$+R$</td>
<td>$-r$</td>
<td>Pericycloid</td>
</tr>
</tbody>
</table>

The following properties are common to all five curves:

1. The normal to any element of the curve passes through the corresponding point of contact of the generating and base circles.
2. The centre of curvature of any element of the curve is at the intersection of the normal with a right line which joins the starting point of the curve with the centre of the base circle.

For the extended, or abridged cycloidal curves the starting point is taken on the radius produced of the curve element, at right angles to a normal to the curve at the point of contact of the rolling circles.

---

Upon the first property depends the suitability of the cycloidal curves for use as tooth outlines, and in the second lies the practicability of approximating them by circular arcs.

### § 256.

**THE GENERATION OF CYCLOIDAL CURVES.**

1. **Exact Solution.** Fig. 574. $G$ is the base circle, $W$ the rolling circle, $A$ the starting point of the curve. Lay off from $A$, on $G$ and $W$, small arcs of uniform spacing, and let $a_1$ and $a_2$ be two of the corresponding points of division. From $A$, with radius $a_1$, strike an arc, and from $a_2$, with the chord $Aa_2$ another arc, and the intersection of the two arcs at $P$, will be a point in the curve.

This solution, which is shown in Fig. 574, both for external and internal rolling, holds good for all five curves.

**Fig. 575.**

2. **Abridged Solution.** From the points $1, 2, 3, \ldots, a$, with radii equal to the corresponding chords of the rolling circle, strike arcs, which arcs will include the entire curve with sufficient accuracy if the points of division be taken sufficiently close together.

In order to draw the extended or abridged curve, starting say at $B$, determine first a point $P_1$ on the ordinary curve, then draw from $a$, with a radius $a_1$, $B$, an arc, and from $P_1$ another with radius $A B$, and the two arcs will intersect in a point $Q_1$ of the curve.

Or, draw through $a_2$, a radius $a_2 b$ in the rolling circle and through $b$, an arc $b C$, concentric with the base circle, and make $a_1 Q_1 = A b$, then will $Q_1$ be the point in the curve for the rolling of the arc $A a_2$ upon $A a_1$.

**Fig. 576.**

### § 257.

**THE GENERATION OF INTERCHANGEABLE TEETH.**

The tooth profile for interchangeable gears is generated in a similar manner, both for external and internal gears, by using a rolling circle of constant diameter for each pitch.

1. **External Teeth.** Fig. 575. Given the number of teeth $Z$, and pitch $t$, or ratio $\frac{1}{n}$ of the wheel. Make $O S = R = \frac{Z t}{\pi}$, and the radius $r_s$ of the rolling circle $W = \frac{t}{2} \cdot \frac{1}{\pi}$. 

---
In the preceding paragraph, the discussion begins with the concept of generating the internal teeth similar to the preceding. The radius of base circle is defined, and the thickness of gear teeth is discussed. The flank circle is described as being generated by rolling a circle upon the base circle, and the face circle is defined. The tooth outline is explained, and the method of calculating the radius of the face and flank circles is introduced. The example calculations are provided for different scenarios, illustrating the application of the formulas.

The text concludes with an explanation of involute teeth for interchangeable gears, emphasizing the importance of their use in practical applications. The diagrams illustrate the concepts discussed, showing the generation of involute teeth and their application in gears.

The diagrams depict the steps involved in generating involute teeth, including the generation of the flank, face, and base circles. The text explains how these circles are used to calculate the necessary dimensions and how they relate to the gear's function. The example calculations demonstrate the application of these principles in real-world scenarios.
the base circle by drawing radii, and the length measured.
For two equal wheels of 14 teeth, ε is only a little greater than unity; it varies between 1 and 2.5.

**Rack Teeth.** Fig. 350. The profile of the rack is straight and makes an angle of 79° with the pitch line T. This angle can readily be laid off by using the drawing triangles of 45° and 30° together.

For low numbered pinions the base circle closely approaches the pitch circle. This sometimes introduces an error into the action. If the portion SB, of the line NN₀, which lies between the pitch and base circles, Fig. 581, is shorter than the length of face of the opposing tooth, this point a will interfere with the flank of the pinion tooth, as shown in the path at c. (See also Fig. 573.) In order to avoid this, the tooth to which the point a belongs must not extend above the line K₀ K₁. This exists for teeth made in the manner given, when Z ≤ 28.

Another method of avoiding this difficulty is to round off the tooth at a, and this is more frequently adopted in practice. An important application of evolute teeth is shown in § 222.

### § 210.

**PIN TEETH.**

Teeth with radial flanks can always be generated by making the inner rolling circle for each wheel equal in diameter to one-half the pitch circle. This will give radial flanks and curved faces to both gears, but wheels made on this system are not interchangeable, and are therefore not practical for general machine construction. Such teeth are still much used by watchmakers on account of the ease with which they may be fitted by filing.

If the diameter of the rolling circle is made greater than the radius of the pitch circle a form of tooth is obtained which is practicable, but which is comparatively little used.

If, in a single pair of wheels, the rolling circle be taken for one wheel equal to the pitch circle, of the other wheel, we obtain for the teeth of the wheel upon which the rolling is done, an outline of cycloidal form, while the teeth of the other wheel become mere points. In practice these points are the centres about which pins are described and such gears are called pinon teeth.

**External Pin-tooth Gearing.** Fig. 582. The pins are circular in section and in diameter equal to 1/2 r₀; the tooth profile for the wheel R₀ is then a curve parallel to the path S₀, described by rolling the circle T₀ on T₀. The arc S₀ = a₀, and circles of the diameter of the pin, struck from successive points of the path S₀, will outline the tooth profile c d, the flank f e being a circular quadrant. The curve of action S₁ is limited by the outer circle K₁₀ at L, and is in all cases greater than L, generally not less than t₁ L. This gives the limit of tooth length a₀ and also determines ε₁. If it is desired to construct the actual line of action, the method of case III, § 209, may be employed.

Fig. 583 shows a pinion of six plus gearing into a wheel of 24 teeth. The diameter of the pin is here made = 1/2 r₀. The flanks of the 24 tooth wheel are made radial with square corners in order to permit ready filing and finishing.

**Internal Pin-tooth Gearing.** Fig. 584. This is similar to the preceding. The tooth profile c d of a parallel to the curve S₁, generated by rolling T upon L, the arc S₁ = a₁, S₁ L is the line of action and is made equal to, or greater than t₁ L. The flank f e is made radial.

In Fig. 585 the pinion is made with the pin teeth and the spur teeth are on the internal gear. The profile of e d is parallel to the curve S₂, generated by rolling T upon L₁; the arc S₂ = a₂, S₂ L₁ is the line of action, as above, and is made equal to, or greater than t₁ L₁; the flank f e is made radial.

In Fig. 586 we make the radius R₁ infinitely great, we obtain a rack, and the tooth profile is a curve parallel to the common cycloid. If we make R₁ = 0 in Fig. 585, infinitely great, we obtain a common form of rack, with pin teeth.

Pin teeth have the practical advantage that they may readily be turned in the lathe. They are especially adapted for situations where they are exposed to the weather, as in sluices, swing bridges, wind-mills, etc. In such cases the pins are often made of round bar iron, without being turned.

**Double Pin Gearing.** Fig. 586. If two gears on this system are run together, one gear may be made with very few teeth, and hence a great difference in velocity ratio obtained, with a minimum distance between centres. In this case both pinon circles become rolling circles. S₁, the pinion face, is generated by rolling T₁ upon T₀, the action extending on S₁ for the point S₁ on the wheel T₁. S₁, the gear tooth face, is generated by rolling T₀ upon T₁, the action extending on the line S₁H₁ for the point S₁ on the wheel T₁. S₁, the flank profile, is made to conform to the theoretical profile S₁ a₁ e₁ (see case IV, § 202), and the other flank is made in a similar manner from the theoretical profile S₁ g₁ e₁. Such gears are sometimes used in hoisting machinery.
THE CONSTRUCTOR.

\section*{Disc Wheels with Pin Teeth.}

It is not an essential requirement that the tooth profile shall be in the immediate line of the pitch circles, as it can be placed within or without to a greater or less extent. In such cases a tooth system is obtained in which the teeth of one wheel pass almost or entirely around those of the other wheel, and hence there can be no so-called backlash, nor is the latter tooth in such a position that the teeth are placed upon the side or face of a disc, or shield, and are called disc wheels, or "shield gearing." *

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig587}
\caption{Fig. 587.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig588}
\caption{Fig. 588.}
\end{figure}

For such wheels pin teeth are well adapted. Fig 587 shows a pair of such wheels arranged for external action, and Fig 588 for internal action. One wheel of each pair is fitted with round pin teeth, and the other has, in the first case, a tooth profile parallel to an extended epitrochoid, and in the second case parallel to an extended hypocycloid.

A peculiar form of disc gearing is shown in Fig. 589. In this case \( R = \frac{3}{4} R' \), \( z = 2, \) and the round pins being on \( R'. \) The flanks of \( R' \) are entirely within the pitch circle, and become straight lines parallel to the straight line hypocycloid \( S' I. \) The arc of action is about \( \pi/4, \) and the backlash can be reduced almost to zero, the teeth on \( R \) being made as rollers.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig589}
\caption{Fig. 590.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig589}
\caption{Fig. 591.}
\end{figure}

Such gear wheels have been described more than once, but are rarely used; they are well adapted to transmit motion to the hands of large tower clocks.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig592}
\caption{Fig. 592.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig593}
\caption{Fig. 593.}
\end{figure}

\section*{Mixed Tooth Outlines. Thumb Teeth.}

By combining the preceding forms of teeth, practical shapes may often be found for special service. The two following examples will illustrate:

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig594}
\caption{Fig. 594.}
\end{figure}

For the low numbered pinions sometimes used in holding machinery, it is important that the pinion teeth shall not be too much undercut, so as to avoid difficulty in making the gears. It is desirable that the flanks on the pinion should be radial. In order to obtain sufficient duration of action, which for a three tooth pinion should not be less than \( 1\frac{1}{2}, \) the face curves of the teeth should be prolonged until they intersect. The curve \( S \) is an arc of an evolute formed by unwrapping the pitch line \( T \) from the circle \( T; \) \( S \) is the radial flank, obtained by rolling the circle \( W \) of radius \( \frac{3}{4} R' \) in \( T; \) \( S, g, \) is the theoretical profile for the tooth space for the wheel \( T. \)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig595}
\caption{Fig. 595.}
\end{figure}

\( S \) acts with the point \( S \) of the rack tooth, and the path \( S H, \) \( S H, \) is a cycloidal curve generated by rolling \( W \) on \( T, \) and acts over the path \( S' I \) with the flank \( S' \) of the wheel \( I. \)

The rack teeth are made straight on the one side, as already shown for rack teeth on the evolute system. Applications for teeth of this form are given in § 226.

§ 213.

TOOTH FRICTION IN SPUR GEARING.

The friction of spur gear teeth is mainly dependent upon the form of the tooth outline, and may be investigated by considering the form, extent and position of the line of action. In most cases the friction is proportional to the duration of action \( c \). A coefficient, dependent upon the position of the line of action may be determined from \( c \), and may be taken as \( \frac{1}{3} \), when the arc of action is equally divided on both sides of the central position; as in the case of epicycloidal teeth, and \( c \) may then, as in many cases, such as pin tooth gearing, the arc of action is entirely on one side of the centre, while for evolute teeth it may be taken as \( \frac{1}{2} \), that being about midway between the two preceding forms. The tooth friction is also greatly dependent upon the number of teeth in both wheels, being proportional to their harmonic mean, and it diminishes rapidly as the number of teeth is increased.

If we make the coefficient of friction \( f \) and take the number of teeth as \( Z \) and \( Z' \), we have for the percentage of loss \( \delta \) in tooth friction:

\[
\delta = f \left( \frac{Z}{Z'} + \frac{Z'}{Z} \right) \frac{\pi}{2}
\]

The value of the coefficient of friction \( f \) is in no case small, even when the teeth are well lubricated, on account of the usual high pressures; a usual value may be taken as \( f = 0.15 \), while for new and dry teeth it reaches 0.30 to 0.35 and even higher.

The minus sign in the formula is to be used when one of the wheels \( (Z) \) is an internal gear.

\[\text{Example 1.} \text{ In a pair of epicycloidal gears, the value of } f = 0.15. \text{ Taking } f \text{ as } 0.15 \text{ we have according to (191 a) for the loss by tooth friction:}
\]

\[f = \frac{0.15 \times 0.15 \times 2 \times 1.99}{0.97} \approx 0.054, \text{ or about 5.4 per cent.}
\]

\[\text{Example 2. Epicycloidal Teeth. } Z = Z', f = 0.15 \text{ and we get:}
\]

\[f = \frac{0.15 \times 0.15 \times 2 \times 1.99}{0.97} \approx 0.054, \text{ or about 5.4 per cent.}
\]

\[\text{Example 3. Epicycloidal Teeth. } Z = Z, f = 0.15 \text{ and we get:}
\]

\[f = \frac{0.15 \times 0.15 \times 2 \times 1.99}{0.97} \approx 0.054, \text{ or about 5.4 per cent.}
\]

\[\text{Example 4. Pin-tooth Gears. } Z = Z, f = 0.15 \text{ and we get:}
\]

\[f = \frac{0.15 \times 0.15 \times 2 \times 1.99}{0.97} \approx 0.054, \text{ or about 5.4 per cent.}
\]

\[\text{Example 5. Evolute Teeth. } Z = Z, f = 0.15 \text{ and we get:}
\]

\[f = \frac{0.15 \times 0.15 \times 2 \times 1.99}{0.97} \approx 0.054, \text{ or about 5.4 per cent.}
\]

\[\text{Example 6. } Z = Z, f = 0.15 \text{ and we get:}
\]

\[f = \frac{0.15 \times 0.15 \times 2 \times 1.99}{0.97} \approx 0.054, \text{ or about 5.4 per cent.}
\]

\[\text{Example 7. } Z = Z, f = 0.15 \text{ and we get:}
\]

\[f = \frac{0.15 \times 0.15 \times 2 \times 1.99}{0.97} \approx 0.054, \text{ or about 5.4 per cent.}
\]

\[\text{Example 8. } Z = Z, f = 0.15 \text{ and we get:}
\]

\[f = \frac{0.15 \times 0.15 \times 2 \times 1.99}{0.97} \approx 0.054, \text{ or about 5.4 per cent.}
\]

\[\text{Example 9. } Z = Z, f = 0.15 \text{ and we get:}
\]

\[f = \frac{0.15 \times 0.15 \times 2 \times 1.99}{0.97} \approx 0.054, \text{ or about 5.4 per cent.}
\]

\[\text{Example 10. } Z = Z, f = 0.15 \text{ and we get:}
\]

\[f = \frac{0.15 \times 0.15 \times 2 \times 1.99}{0.97} \approx 0.054, \text{ or about 5.4 per cent.}
\]

\[\text{Example 11. } Z = Z, f = 0.15 \text{ and we get:}
\]

\[f = \frac{0.15 \times 0.15 \times 2 \times 1.99}{0.97} \approx 0.054, \text{ or about 5.4 per cent.}
\]

\[\text{Example 12. } Z = Z, f = 0.15 \text{ and we get:}
\]

\[f = \frac{0.15 \times 0.15 \times 2 \times 1.99}{0.97} \approx 0.054, \text{ or about 5.4 per cent.}
\]

\[\text{Example 13. } Z = Z, f = 0.15 \text{ and we get:}
\]

\[f = \frac{0.15 \times 0.15 \times 2 \times 1.99}{0.97} \approx 0.054, \text{ or about 5.4 per cent.}
\]

\[\text{Example 14. } Z = Z, f = 0.15 \text{ and we get:}
\]

\[f = \frac{0.15 \times 0.15 \times 2 \times 1.99}{0.97} \approx 0.054, \text{ or about 5.4 per cent.}
\]

It will be seen that the tooth friction is least with epicycloidal teeth and greatest for pin gearing; evolute teeth being midway between.

The wear upon gear teeth is affected by other considerations besides that of the coefficient of friction, the pressure of the teeth upon each other, and the relative rubbing movement of various portions of the profile also entering into the problem. The wear is therefore not constant for a constant pressure, and it is an error to assume, as is sometimes done, that the form of evolute teeth is unaltered by wear. These teeth usually show the greatest proportional alteration by wear, since the flank of the tooth below the pitch circle has a very much less rubbing movement than the portion of the opposite tooth which rubs against it and hence the wear is unequal.

* This form of mixed outline has been described by Will in 1874; it was revived by him in 1874 and used in practice; he made the angle a greater than here given, viz. 0°.
The effect of this may frequently be observed in practice, where the smaller of a pair of evolute gear wheels will be noticed to be worn into deep hollows below the pitch circle.

The power above shown about the percentage of loss may also be determined geometrically in the following manner:

Take the two portions of the tooth profile which work together and divide each by the chord of the corresponding portion of the line of action, multiply each result by the ratio of the length of its portion of the line of action to the entire length of the line of action, and then multiply the sum of the two quantities by the coefficient of friction.

The result will be the percentage of loss, $p$. The chord referred to becomes the line of action itself in the case of evolute teeth. This method serves also for pin teeth, and is very useful for the designer, as the data can all be taken off the drawing with the dividers.

§ 214.

GENERAL REMARKS ON THE FOREGOING METHODS.

Each of the preceding methods possesses its merits and disadvantages.

Epicycloidal Teeth. These possess the great advantage that they will work together in any series with as few as seven teeth, while for evolute teeth the lowest in series is 14 teeth, and in no case fewer than 11. The loss from tooth friction is a minimum with this form, and the wear less injurious to the shape of the tooth. The minor objections which have been raised are that the double curve increases the difficulty of construction, and that any variation of the distance between centres causes imperfect action to follow.

Evolute Teeth. The advantages of this form are that the simple shape is readily made and that any variation of the distance between centres does not affect the action.

Against these must be set the fact that for low numbered pinions the flanks must be altered to avoid interference, or the tops of the teeth must be taken off. The fact that the distance between centres may vary is rather an objection in many cases, as the arc of action is reduced, and in transmission of heavy power the shocks upon the teeth are liable to be increased. Evolute teeth are well suited for interchangeable gears, if low numbered pinions are not required (30 teeth being the minimum), and where but small power is to be transmitted they are excellently adapted. For wheels which run only in pairs, and hence for bevel gears, this form is excellent, since it is so readily made. See § 212.

Pin tooth gearing and the mixed outlines are only used for special work, such as in hoisting machinery and the like, and in such cases the wheels are often made of wrought iron or steel. Disc wheels have a very limited application, but in some special forms of mechanism they are very useful, and will be discussed further. See Chapter XVIII.

§ 215.

B. CONICAL GEAR WHEELS.

GENERAL CONSIDERATIONS.

In the case of conical gear wheels, or as they are generally termed, bevel gears, the working circles of a pair of gears which run together, lie on the surfaces of a pair of cones, the apex of each cone being at the intersection of the axes of rotation. In such cases the pitch circles are taken at the $p$ base circles of the respective cones, as $SD$, and $SE$, Fig. 596. The length of the teeth is measured on the supplementary cone, to each base cone, $SB$ being the supplementary cone for $SD$, and $SC$ that for $SE$. If $CB$ be taken at right angles to $AS$, the length of teeth is laid off on $SA$, and the width of face on $SA$: the tooth thickness being spaced off on the pitch circle and all the teeth considered to the point $A$.

The respective radii $SD$ and $SE$ of the two cones are found by dividing the angle $a$ of the axes, in such a manner that the perpendiculars $SD$ and $SE$ let fall from $S$ to the axes, bear the same ratio as the numbers of teeth, or inversely as the number of revolutions; thus $SD:SE = Z:Z$, $n_1 = n_2$. There are, therefore, two solutions possible, according as the pitch line $SA$ is taken within the angle $a$, or in its supplement; or which is the same thing, according to which angle is taken as the angle of the axes. The difference between the two consists in the fact that for a constant direction of revolutions the driving shaft the driven gear revolves in one direction for the first solution and in the opposite direction for the second solution. One of the solutions gives an internal gear, when $n_1 > n_2 \cos a$.

If bevel gears are required to interengage (see § 200) they must not only be of the same pitch, but must also have the same length of contact line, $AS$, Fig. 596. Since these conditions are very infrequent, it follows that bevel gears are generally only made to work in pairs. In practice it is found that a variation of less than 5 per cent. in the length of the contact line may be neglected. Gears of the same pitch and same angle of axes, but with a small variation of contact line, are called "baisard gears." A pair of right angle bevel gears of 80 and 45 teeth, might be altered in practice, if required, into biasard gears of 80 ($1 = 0.95$), e.g., 81 to 76 teeth, which would work with the other gear of 45 teeth.

§ 216.

CONSTRUCTION CIRCLES FOR BEVEL GEARS.

The geometrical figures which are formed by one cone rolling upon another, require that both cones should have a common apex. The surface thus developed is called a spherical cycloid. Of these there are five particular forms, as with the plane cycloids, the latter being really those for a cone with an apex angle of 180°. The spherical cycloid is very similar in form to the plane cycloid, as are also the corresponding evolutes; the branches of the curves assuming a zig-zag form.\(^6\)

\[\text{FIG. 597.}\]

\[\text{FIG. 596.}\]

\[\text{FIG. 597.}\]

The use of the spherical cycloid for the formation of bevel gear teeth would involve many difficulties. In order to construct such teeth, it is therefore common to use the method (first devised by Trevid) of auxiliary circles, based upon the supplementary cones, and enabling the teeth to be laid out in a similar manner to those of spur gears. The auxiliary circles for the bevel gears $R$ and $R_1$, Fig. 597, are those of the spur gears having the same pitch, their radii being respectively $r$ and $r_1$, the elements $BS$ and $CS$ of the supplementary cones.

For any given angle $a$ between the axes, the radius $r$, and number of teeth $3$, for the auxiliary circle can be determined

\[\text{See Berliner Verhandlungen, 1876, pp. 301, 406. Reuleaux, Development of the Spherical Cycloid.}\]
from the radii \( R \) and \( R_0 \), and tooth numbers \( Z \) and \( Z_1 \), by the following formula:

\[
\begin{align*}
R &= \sqrt{R^2 + R_0^2 + 2R R_0 \cos a} \\
K &= \frac{R_1}{R} \\
Z &= \frac{Z_1}{Z_1} + 2Z \cos a \\
Z &= \frac{Z_1}{Z_1} + 2Z \cos a
\end{align*}
\]

If the axes are at right angles, we have

\[
\begin{align*}
K &= \frac{\sqrt{R^2 + R_0^2}}{R_1} \\
Z &= \frac{Z_1}{Z_1} \\
R &= \left( \frac{\sin a}{\sin b} \right)^{\frac{1}{2}}
\end{align*}
\]

Example—A pair of bevel gears have 30 and 40 teeth, and an angle between axes \( a = 60^\circ \), hence \( \cos a = \frac{1}{2} \), and we have for the auxiliary circle of the 30 tooth gear:

\[
z = \frac{30}{2} + 30 \times \frac{1}{2} = 45
\]

For the 40 tooth gear we have also:

\[
z = \frac{40}{2} + 40 \times \frac{1}{2} = 60
\]

From these numbers and the given pitch, the auxiliary circles can be laid off and the tooth drawn.

Low tooth numbers are not available for bevel gears, since the errors which are involved in the method of auxiliary circles become disproportionately great. By using not fewer than 24 teeth for the bevel gear, a minimum of 28 for the auxiliary circle is obtained, and the evolute system can be used to advantage. This form of tooth is best adapted for this purpose, on account of its simplicity of form, notwithstanding the minor defects which have already been noticed.

The loss from tooth friction in bevel gears is approximately equal to that of their corresponding auxiliary gears.

\[ \text{Fig. 598.} \]

\[ \text{§ 217.} \]

The Plane Gear Wheel.

Internally toothed bevel gears are not used, on account of the practical difficulties involved in their construction. There is, however, an interesting form of gear wheel which lies intermediate between the external and internal forms. If the numerical ratio between a pair of bevel gears is \( \frac{R_1}{R} \), one of the solutions for the base cone gives the latter a plane surface, \( S \).

\[ \text{Fig. 599.} \]

The supplementary cone in the case becomes a cylinder, and the radius of the construction circle becomes infinitely great, hence the tooth outlines are similar to those used for rack teeth. If the evolute system is used the teeth are very simple, and the plane gear in some cases becomes a very convenient form of construction.

As already stated, the ratio is

\[
\frac{R_1}{R} = \cos a 
\]  

\[ (193) \]

from which, if for example \( a = 60^\circ \), we have \( \frac{R_1}{R} = \frac{1}{2} \). If the angular relation of the axes is given it follows that but one velocity ratio can be obtained. This is determined from the angle \( \gamma \), which is one-half the apex angle of the cone \( R_0 \) and from the ratio \( \frac{R_1}{R} = \sin \gamma \).

It is sometimes very convenient to arrange a plane gear so that it may work with both of a pair of bevel wheels. This is shown in Fig. 599, in which the gears \( R_1, R_2 \) have the semi-apex angles \( \gamma \), \( \gamma_2 \), and have their axes at right angles. We then have:

\[
R_1 = \tan \gamma = \cot \gamma_2
\]

from which we obtain the following values:

\[
\begin{align*}
R_1 &= \tan \gamma = 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 2 \quad 3 \quad 4 \\
\gamma &= 90^\circ, 89.9^\circ, 89.8^\circ, 89.7^\circ, 89.6^\circ, 89.5^\circ, 89.4^\circ, 89.3^\circ \\
\gamma_2 &= 89.4^\circ, 89.5^\circ, 89.6^\circ, 89.7^\circ, 89.8^\circ, 89.9^\circ, 90^\circ \\
\sin \gamma &= 0.2422, 0.3177, 0.4499, 0.6000, 0.7677, 0.8990, 0.9948, 1.0000
\end{align*}
\]

Either of the wheels \( R_1, R_2 \) can be used with the plane gear \( R_0 \) if the number of teeth have the ratio given by the value of \( \sin \gamma \). Although this limits its application, yet the plane gear is frequently found very useful for angular transmissions.

C. HYPERBOLODIG GEAR WHEELS.

§ 218.

Base Figures for Hyperboloidal Wheels.

Hyperboloidal wheels are used to transmit motion between inclined, non-intersecting axes. The figures upon which they are based are hyperboloids of revolution having a common generatrix. These may be determined in the following manner.

\[ \text{Fig. 600.} \]

In Fig. 600 is given a projection normal to the line of shortest distance between the two axes. The angle \( a \) is divided into two parts \( \beta \) and \( \gamma \), in such a manner that the perpendiculars let fall from any point \( A \) of the line \( S \), upon the two axes, shall be inversely proportional to the revolutions of the gears. \( S, A \) is then the contact line of the hyperboloids; \( A \) is perpendicular to the axes.

* The so-called "Universal Gear" of Prof. Dietrich, introduced in 1868, should be considered as a variety of conical gear in which the angle of the axes may be conveniently varied. These may be used for axes of angles varying from 0° to 90°. As shown in the illustration, these wheels are formed of globoids of the 11th class (see § 244), the meridians forming the teeth and spaces. They have found but limited application. A model of these gears is in the kinematic cabinets of the Royal Technical High School.
THE CONSTRUCTOR.

\[ R' = \frac{\sin \beta}{\sin \beta_1} = \frac{n_1}{n} = \frac{Z}{Z_1} \tag{194} \]

The actual radii \( R \) and \( R' \) are yet to be determined, as well as the radii \( S \) of the hyperboloids intersecting at \( A \). We have:

\[ \frac{r}{r_1} = \frac{\tan \beta}{\tan \beta_1} = \frac{n_1}{n} \cdot \frac{\cos \alpha}{\cos \alpha} = \frac{n_1}{n} \tag{195} \]

that is, \( r \) and \( r_1 \) have the same relation to each other as the portions \( \Delta \) and \( \Delta' \) of the line of contact. If we call the shortest perpendicular distance between the axes \( a \), we have:

\[ \frac{r}{a} = \frac{1 + \frac{n}{n_1} \cos \alpha}{1 + 2 \frac{n}{n_1} \cos \alpha + \left( \frac{n}{n_1} \right)^2} \tag{196} \]

\[ \frac{r_1}{a} = \frac{1 + \frac{n}{n_1} \cos \alpha}{1 + 2 \frac{n}{n_1} \cos \alpha + \left( \frac{n}{n_1} \right)^2} \]

The radii \( R \) and \( R' \) are hypotenuse for the triangles whose sides are \( R' \) and \( r, R' \) and \( r_1 \) (see the left of the figure) or:

\[ R = \sqrt{R'^2 + r^2} \]

\[ R_1 = \sqrt{R'^2 + r_1^2} \tag{197} \]

\( R' \) and \( R_1' \) being determined as above, when the distance \( S \Delta \) is given. For the angles \( \beta \) and \( \beta_1 \) we have the general expressions:

\[ \tan \beta = \frac{\sin \alpha}{n_1 + \cos \alpha} \tag{198} \]

\[ \tan \beta_1 = \frac{\sin \alpha}{n + \cos \alpha} \]

As in the case of bevel gears, two solutions are possible according as the angle \( \alpha \), or its supplement, is taken in determining the line of contact \( S \Delta \) Fig. 601. The choice of solution also:

\[ \frac{r}{a} = \frac{n_1^2}{n^2 + n_1^2} \tag{200} \]

\[ \frac{r_1}{a} = \frac{n^2}{n^2 + n_1^2} \]

In the construction of the wheels, corresponding zones are chosen on the two hyperboloids. If the distance between the axes is small, the zones lying in the hyperboloid zones are generally unsuitable, but when the distance is greater they may be used and the figures approximated by truncated cones.

![FIG. 602.](image)

**FIG. 601.**

![Diagram](image)

**FIG. 602.**

*Example 1.* \( a = 45^\circ, \frac{n_1}{n} = \frac{1}{2} \). (See Example 1, in § 222). If \( a = 45^\circ \).

We have:

\[ R = \frac{a_1 + \cos \alpha}{1 + \cos \alpha} = \frac{2.656}{1 + 0.432} \]

\[ R_1 = \frac{1 + \cos \alpha}{1 + \cos \alpha} = \frac{2.656}{1 + 0.432} \]

Also \( \tan \beta = \frac{\sin \alpha}{n_1 + \cos \alpha} \). If \( a = 45^\circ \).

\[ \frac{r}{a} = \frac{n_1}{n} \cdot \frac{\cos \alpha}{\cos \alpha} \]

\[ \frac{r_1}{a} = \frac{n_1}{n} \cdot \frac{\cos \alpha}{\cos \alpha} \]

\[ \tan \beta_1 = \frac{\sin \alpha}{n + \cos \alpha} \]

\[ \tan \beta_1 = \frac{\sin \alpha}{n + \cos \alpha} \]

For the \( a = 45^\circ \), have tan \( \beta = \frac{\sin \alpha}{n_1 + \cos \alpha} \). Hence \( \beta = 60^\circ, 57^\circ \), and \( R_1 = 69^\circ \).

If we make \( a = 45^\circ \), we have from (199):

\[ \frac{r}{a} = \frac{n_1}{n} \cdot \frac{\cos \alpha}{\cos \alpha} \]

\[ \frac{r_1}{a} = \frac{n_1}{n} \cdot \frac{\cos \alpha}{\cos \alpha} \]

\[ \tan \beta_1 = \frac{\sin \alpha}{n + \cos \alpha} \]

\[ \tan \beta_1 = \frac{\sin \alpha}{n + \cos \alpha} \]

For the \( a = 45^\circ \), have tan \( \beta = \frac{\sin \alpha}{n_1 + \cos \alpha} \). Hence \( \beta = 60^\circ, 57^\circ \).

The appearance of such a pair of gears is shown in Fig. 601. According to the table in § 222 the pitch for the larger gear is \( \frac{z}{z_1} = \frac{60}{57} = 1.056 \), and for the smaller gear \( \frac{z}{z_1} = \frac{60}{57} = 1.056 \).

**Example 2.** \( a = 90^\circ, \frac{n_1}{n} = \frac{1}{2} \). or say the number of teeth \( z = 36 \), and \( R_1 = 20 \); \( a = 90^\circ \). We have from (199):

\[ \frac{r}{a} = \frac{n_1}{n} \cdot \frac{\cos \alpha}{\cos \alpha} \]

\[ \frac{r_1}{a} = \frac{n_1}{n} \cdot \frac{\cos \alpha}{\cos \alpha} \]

\[ \tan \beta_1 = \frac{\sin \alpha}{n + \cos \alpha} \]

\[ \tan \beta_1 = \frac{\sin \alpha}{n + \cos \alpha} \]

For the \( a = 90^\circ \), have tan \( \beta = \frac{\sin \alpha}{n_1 + \cos \alpha} \). Hence \( \beta = 90^\circ, 57^\circ \).

**Example 3.** \( a = 90^\circ, \frac{n_1}{n} = \frac{1}{2} \). or say the number of teeth \( z = 36 \), and \( R_1 = 90^\circ \). In this case the hyperboloids become similar (see Example 4, § 222).

**Example 4.** In the special case in which \( a = 90^\circ \), and the position of the contact line, which is determined by \( \beta \), lies in the supplement to \( a \), so that \( \beta = 90^\circ \), the base figures become the one normal cone and the other a plane hyperboloid, see Fig. 605. This construction is similar to the preceding forms of plane and bevel gears, and may be conveniently used to work with a train of common bevel gears, although but few practical applications occur, partially owing to the fact that the prolonging axis of the bevel
If it is desired to approximate to the hyperboloidal zone by the use of a conical surface, the apex must be determined. In this case the generatrix \( S \cdot A \) is rotated about the axis \( H \cdot S \) until \( A \) falls on the point \( J \) of the circumference, when the new projection of the generatrix will pass through the apex \( M \) of the cone.

The tooth friction of hyperboloidal gears is necessarily great. This will be considered later, in connection with the speed of the rubbing surfaces, which is similar to that of the spiral gears, which are tangent at the gorge circles (606 § 220).

**D. SPIRAL GEARS.**

**CYLINDRICAL SPIRAL GEARS.**

Cylindrical spiral gears may be used in the same manner as hyperboloidal gears for the transmission of motion between inclined axes, and in some cases possess advantages over the latter. There are a number of useful variations of spiral gears.

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**TEETH FOR HYPERBOLOIDAL GEARS.**

The construction of the exact forms for the teeth of hyperboloidal gears is a very difficult operation, and in practice an approximation is used similar to that employed for bevel gears. The method adopted is to determine the supplementary cone to the hyperboloid used, and as in the case of bevel gears, use the corresponding construction circle.

The apex \( H \) (Fig. 606) is determined by drawing \( A \cdot H \) perpendicular to the generatrix \( S \cdot A \), which, as before, is taken parallel to the plane of the drawing. The teeth will be formed with sufficient accuracy if two construction hyperboloids are taken with the same angle of contact as the base hyperboloids, according to the conditions in (108) and (109), and the teeth are formed on the surfaces, which are described by the edges of the construction hyperboloids upon the base hyperboloid.

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For \( \alpha = 90^\circ \) we have \( \cot \gamma = \frac{m}{n} \). Such spiral wheels, when the teeth are well made, transmit motion very smoothly, but the surface of working contact is very small. When the axes are at right angles and the wheels the same size, it is often inconvenient to use spiral gears on account of the large size required.

\[ \text{Fig. 607.} \]

Example. Fig. 607. Let \( \frac{m}{n} = 2 \) and \( \alpha = 90^\circ \). We have from (100) \( \frac{R_1}{R} = 2 \), \( \frac{m}{n} = 2 \), \( \frac{n}{m} = 1 \), whence \( \gamma = \frac{\pi}{2} - \phi \), and \( \phi = \frac{\pi}{4} \). The sliding velocity is \( v_s = c (\phi + \phi) \sqrt{2} \). The small value of the angle \( \gamma \) makes it undesirable to use the smaller gear as the driver. These objectionable features are of increasing importance and for example, when \( \frac{m}{n} = 2 \) and \( \frac{n}{m} = 10 \), we get \( \gamma = 0^\circ \), and \( \phi \) about \( \frac{\pi}{2} \). The difficulty of cutting the teeth on the latter also increases, as may readily be seen.

\[ \text{Sec. 221.} \]

APPROXIMATELY CYLINDRICAL SPIRAL GEARS.

If, of the preceding conditions, only those of formula (201) and (202) are strictly observed, the difficulties of construction are much reduced and at the same time satisfactory wheels obtained.

Three methods may be employed: (a) a slight modification from the correct spiral form may be given to both wheels, (b) one gear may be made a true spiral, and the variation all thrown into the other gear, or (c) the wear which is at first caused by running the approximate forms together may be disregarded until the parts have worn themselves into smooth action. From these reasons a widely varying pitch in the construction of spiral gears will be found. One of the most important applications is that of the worm and worm wheel, Fig. 608. In this case \( \alpha = 90^\circ \) and \( \frac{n}{m} = 1 \), the teeth of the wheel \( R_1 \) being inclined at an angle \( \gamma \) with the edge of the wheel, whence tan \( \gamma = \frac{2 \pi R}{m} \).

\[ \text{Fig. 608.} \]

It is arranged shown in Fig. 609, we have \( \alpha = 90^\circ \) and \( \gamma \) and the teeth on \( R_1 \) are made parallel to the axis. The pitch of the screw is here made \( \frac{m}{n} \) for a pitch \( t \) of the wheel. The velocity ratio of transmission, from the fundamental formula (180) is \( n_1 : n_2 = \frac{Z_1}{Z_2} \); or this case it equals \( \frac{Z_1}{Z_2} \).

In many cases the worm is made a true spiral and the consequent wear disregarded, but in more careful work the method (a) is adopted and the worm wheel cut with a hob, which makes the proper modification in the shape of the teeth.

The friction between the worm and teeth of the worm wheel is very great, as the thread slides entirely across the teeth. We have for the coefficient of friction \( \mu \) for the ratio between the actual force \( F' \) and a force \( F \) acting at the same lever arm on the screw, but free from frictional resistance, approximately:

\[ F' = \frac{1}{\mu} \cdot \frac{z \times R}{R} \]

For \( \mu = 0.16 \) we have practically:

\[ F' = \frac{1}{\mu} \cdot \frac{R}{L} \cdot \text{etc.} \]

It follows that to obtain the minimum of frictional loss, \( \frac{R}{L} \) must be made as small as practicable.

Morin gives the rule \( R = t \), which makes \( F' = 4 \); Redtenbacher makes \( R = 1.5 t \), whence \( F' = 2.6 \). If we make \( R = t \), we get \( F' = 2 \), and this as low as \( R \) can well be made. In this case it will be seen that a higher efficiency than 50 per cent. cannot be obtained, and it is also apparent that the worm must be the driver, since the resistance of friction would just balance the reverse driving action. The ordinary tooth friction and the journal friction must of course be added.

\[ \text{Fig. 610.} \]

\[ \text{Fig. 611.} \]

\[ \text{Fig. 612.} \]

The tooth outlines for both worm and wheel are the same as for a rack and gear wheel, taken on a longitudinal section through the axis of the worm. The evolute tooth is especially applicable, and \( Z_1 \) must not be less than 25 (209). The surface of contact is theoretically only a mathematical point, but in practice there is a small flattened surface of contact, and if a larger surface is desired the wheel must be cut with a hob of the same form as the worm which is to work with it.

Wheels which have a contact bearing of a point only, may be called precision gears, as distinguished from power-transmitting gears. The difference, however, cannot be sharply maintained, for as already shown, worm gearing is used for the transmission of both large and small forces.

The possible variations of the pitch angle permit a great variety of spiral gear combinations, as the following examples show:

Example 1. Given \( \gamma = \frac{\pi}{4} \), the perpendicular distance between axes \( a = R + R_1 \) and the angle between axes \( a = 45^\circ \). If we make \( \gamma = 45^\circ \), we have from (207) \( \gamma = 45^\circ + 45^\circ - R - R_1 \). Hence (see Fig. 610), and from (100) \( R_1 = \frac{60^\circ \gamma R}{60^\circ \gamma} - \frac{1}{4} \). The sliding velocity is \( v_s = \frac{1}{2} \frac{60^\circ \gamma R}{60^\circ \gamma} - \frac{1}{4} \). The sliding velocity is \( v_s = \frac{1}{2} \frac{60^\circ \gamma R}{60^\circ \gamma} - \frac{1}{4} \). From which \( R_1 \) and \( \gamma \) may be readily determined. If we make \( \gamma = 45^\circ \), we have:

\[ R_1 = \frac{60^\circ \gamma R}{60^\circ \gamma} - \frac{1}{4} \]

and \( R = 145^\circ \). For \( \gamma = 45^\circ \), the normal pitch \( t = \sin \gamma = 1 + \frac{R_1}{R} \). The circumferential pitch \( t' = \frac{R_1}{R} - \frac{1}{2} \). The sliding velocity is \( v'_s = \frac{1}{2} \frac{60^\circ \gamma R}{60^\circ \gamma} - \frac{1}{4} \). The sliding velocity is \( v'_s = \frac{1}{2} \frac{60^\circ \gamma R}{60^\circ \gamma} - \frac{1}{4} \). The sliding velocity is \( v'_s = \frac{1}{2} \frac{60^\circ \gamma R}{60^\circ \gamma} - \frac{1}{4} \). The sliding velocity is \( v'_s = \frac{1}{2} \frac{60^\circ \gamma R}{60^\circ \gamma} - \frac{1}{4} \).

In Fig. 611, we have \( \gamma = \frac{2}{3} \) and \( R_1 = \frac{5}{3} R \), whence \( R = 133.3^\circ \). It is seen that the value of \( R' \) in Example 1 approached very closely to the minimum.

*In the illustration \( Z_1 = 4 \), which in (207) for a true spiral would require \( R_1 = 900 R \), and \( \gamma = 85.2^\circ \).
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Example 9. If so desired we may make \( \gamma = 90^\circ \), when one wheel will become an ordinary spur gear, Fig. 612, and we have \( \gamma = 0 \), and \( \gamma_1 = 90^\circ \), and \( R = 0 \). The two wheels are similar, both in the right hand. The sliding velocity is \( c = v \cos \gamma \). In the arrangement shown in Fig. 614 there is added to the right hand, Fig. 615, when the wheels are placed in the right hand, the right hand will revolve in the opposite direction.

The middle gear reverses the motion. In this case the gear is in the middle of the bevel gears.

Example 10. When \( a = b \) and \( \gamma = 90^\circ \), the wheel in the middle of the bevel gears.

Spiral Gears are cut in a similar manner to screws, the tool being carried in the slide rest of an engine lathe, and set at the proper angle. The pitch of the screw thread is:

\[ \rho = 2 \pi R \tan \gamma \]

and the travel of the rest is effected by proper change gears, according to the selected values of \( \gamma \) and \( \gamma_1 \).

The tooth outline to be used is determined according to the radius of curvature of the spiral, that is, to that at right angles to the spiral to be cut. The radius of curvature \( r \) and \( \rho \) are

\[ r = \frac{R}{\sin \gamma} \quad \rho = \frac{R}{\sin \gamma_1} \]

These give the radii for the construction circles to be used with the pitch \( \rho \); the shape of the tool with which the teeth are cut is then determined.

Example 11. For the wheel of the first example in the preceding section, we have:

\[ r = \frac{R}{\sin \gamma} \quad \rho = \frac{R}{\sin \gamma_1} \]

If it is preferred to determine \( r \), graphically from formula (208) the method given in 22 is employed.

The frictional resistance of spiral gears is often a matter of much importance. If the frictional resistance is assumed to be zero, we have for the relation of the force \( P \) applied to the driving wheel, to the force \( Q \) delivered by the driven wheel:

\[ P = \frac{\sin \gamma}{\sin \gamma_1} \]

The ordinary tooth friction, which is the same as that of the construction gears (see 211) to which must be added the frictional force to the sliding of the teeth, whenever it is greater than zero. The value of the latter friction is governed by the sliding velocity \( \nu \). For the calculation of the loss of useful effect we may use the formula:

\[ \frac{P}{P} = \frac{\sin \gamma}{\sin \gamma_1} \sin (\gamma + \rho) \]

in which \( \gamma \) is the angle of friction for the coefficient \( f \), whence \( \tan \gamma = f \) for \( f = 0 \), we have \( \gamma = 0 \).

Example 12. For the wheels in the preceding example we have:

\[ \frac{P}{P} = \frac{\sin \gamma}{\sin \gamma_1} \sin (\gamma + \rho) \]

To this must be added the ordinary friction of the equivalent spur gears. Another source of loss is that due to the lateral forces \( K \) and \( K_1 \), acting in the direction of the axes. For these we have:

\[ \frac{P}{P} = \frac{\cot (\gamma + \rho)}{\cot (\gamma - \rho)} \]

Example 13. For the preceding gears we have:

\[ \frac{P}{P} = \frac{\sin \gamma}{\sin \gamma_1} \sin (\gamma + \rho) \]

\[ \frac{P}{P} = \frac{\cot (\gamma + \rho)}{\cot (\gamma - \rho)} \]

* Bredt's Tables will be found of service in arranging change gears. (Calcul des Rouages par Approximation, Paris, 1869).
The tooth friction may be reduced to a very small amount by reducing the bearing surface of the teeth of one gear to a point of contact, or practically to a knife edge. Such gears (devised by Hooke) are only of use for purposes of precision, but in some cases are found serviceable.*

The space between teeth at the middle of the gear, is called in the Westphalian shape the "spring" of the teeth. If it is desired to approximate to the frictionless action of the teeth, this "spring" must be slightly greater than the pitch.

For very large transmissions the gears may be made in two parts. Fig. 628 shows a pair of such gears for a reversing rolling mill by the Haggert Steel Works. The pitch diameter is 43.5", the pitch 8°30', the face of each gear 20°, and the total weight 24,000 pounds. The teeth are made with double reverse angles on each gear, so that the conditions are the same when running in either direction, and the whole is a masterpiece of machine work in steel.

A similar form to the preceding gears is the so-called step-gearing, Fig. 626, frequently used in planing machines (by Shank, Collier and others). The tooth profiles may be modified as above, to reduce friction, but the graduati should be as great or greater than the pitch. Fewer teeth should not be used.

A spiral bevel gear, as in spur gears, and in this case the distance between the axes becomes zero, while the angle of the teeth is at least 4. For the curvature of the tooth it is best to use a conical spiral of constant pitch, the projection of which on the base of the cone is an Archimedean spiral. Frequent applications of such wheels are to be found in spinning machinery, and they are operated successfully at quite high velocities.*

* For a machine for the correct construction of the teeth of spiral bevel gears, see Genic Industriel, Vol. XII, p. 385.
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The second class gives the inclined globoid, Fig. 635, with unsymmetrical conical sections, with regard to the equator, the spiral being on the zone mantles. If $\delta = 90^\circ$ we obtain a symmetrical, cylindrical hollow section of a sphere, Fig. 636. The spiral, when $a = a$, becomes a spherical cycloid. If $\delta = 90^\circ$ the figure becomes a plane cone, or plane ring; and the curve becomes a plane cycloid.

We have the third class when $\delta = 0$, and $a > r$, giving a so-called cylindrical ring, Fig. 637. $r$, and when $a < r$, the apple shaped globoid, Fig. 637.

If $\delta$ is an acute angle, the globoid is flattened, Fig. 638; the globoid of Class I is the limiting case. The spiral curves are globoidal cycloids, which become plane figures when $\delta = 90^\circ$, and the globoid becomes a plane ring or plane cone.

The fourth class gives the highest forms, Fig. 639, in which $\delta = 0$, and we may have $a > r$, $d = r$, or $a < r$.

The inclined globoids of this class have forms, the limits of which are found in those of the second class, Fig. 635. If $\delta = 90^\circ$ we have again the plane cone or plane ring.

The practical applications of the globoid spiral gears are varied, and are found mainly in right globoids of classes III and IV.

* Two right globoidal rings may unite to form a pair of machine elements when the thickness of one is made equal to the hole in the other, as in Fig. a. The two parts then bear the relation to each other of journal and bearing, and are similar to a ball joint. Each of the two elements describes by the relative motion of any point a corresponding path on the other member.

These conditions are approximately found in a pair of chain links. Such a pair may also be considered a contracted form of universal joint, $\phi \phi \phi$. Fig. $\delta$, the same relative motion existing between $\phi \phi \phi$ and $\phi \phi \phi$. The same thing is shown in a fractional form in Fig. a, where some method of holding the parts together, such as bands, etc., must be used. This latter resembles closely the ball and socket joints of the human skeleton.
IV. In the valve gearing of Stephenson’s locomotives, Fig. 640, is found a globoid worm of class IV, using the middle part of the globoid form. Fig. 637, $a$ ($a < r$). In this case, the reversing lever $K$ is really a part of an internal gear with a radius $r_1$, the radius of the describing circle. In this case, the internal gear has but a single tooth, although more might be used.

![Diagram](image1)

It will be seen that the globoid forms can be used as internal gears. This is shown in Fig. 641, which represents a worm formed as a globoid screw. Its form is practically the same as that of the hole in the right globoid ring, Fig. 637 $a$. The section shown in the figure is of such length that it includes one-fourth of the entire circumference of the worm wheel $B$, although it could be extended so as to include almost one-half.

![Diagram](image2)

The most important point to be considered is the formation of the teeth. $R_1$ is again made equal to $r$. Since the globoid is used in the internal form, the two tooth profiles, on $r$ and $A_1$, fall together. The sliding is in the plane of a normal section through $C$ and $A_1$, and not endwise, and hence the shape of the teeth is absolute.

![Diagram](image3)

(The internal gear tooth, with $R = R_1$.) The teeth can be made of straight profile in the worm wheel as well as in the worm. The production of the globoid worm in the lathe is not difficult. This form has been frequently used in recent work. The advantages appear to be in the simple form of tooth and in the completeness of the engagement.

---

An interesting modification is that of Hawkins, Fig. 642$.^*$ In this case, the wheel $B$ is composed of friction rollers of quite large size and the friction is thereby greatly reduced. Instead of there being only four teeth, as would at first appear, there is in reality an ideal number of teeth, a condition referred to in the fundamental discussion in § 200. If for every revolution of the globoid screw, one tooth of the wheel engages, there must for each space formed between the rollers be 10 teeth to a quarter revolution, so that instead of 4 teeth in $B$, there are $4 \left(1 + \frac{1}{10}\right) = 4.4$ teeth.

![Diagram](image4)

The gearing used in Jensen’s Winch, Fig. 643, belongs to the globoid class IV, of the form shown in Fig. 639. Usually in this form $a = r$, although sometimes $a < r$, as in Fig. 639 $a$. $R_1$ is again made $= r$, and the internal globoid form used. The ratio is so chosen that a slow motion can be converted into a fast one, as may also be done with the form shown in Fig. 641 if the pitch of the worm is made sufficiently great. The use of rollers instead of teeth makes a very satisfactory construction.$^+$

---


If in the first two classes of globoids the supplementary axis is removed an indefinite distance, the globoids become plane surfaces, and the globoid screws whereby reach the limit. The limiting case of Class III is the ordinary worm and worm wheel, and another form is Long's spiral gearing, which also belongs to Class III; α is chosen at will, c = d = o. The globoid becomes a plane cone, and the globoid screw becomes an Archimedean spiral. If K becomes indefinitely great, we obtain a disk with a spiral groove engaging with a rack, the middle section having full tooth contact from top to bottom.† When this is brought into Class IV, we obtain the Archimedean spiral in its most general form, i.e., the evolute of a circle.

E. CALCULATION OF PITCH AND FACE OF GEARING.

§ 225.
PITCH OF GEAR WHEELS. TOOTH SECTION.

The dimensions of gear wheels must, for the same pressure on the teeth, be increased to meet shock in proportion to the increase in initial velocity. For slow running gears this action can be neglected. We may in this respect, therefore, divide gears into two classes, viz.:

1. Hoisting Gears and Transmission Gears, and includes under the term hoisting gears all those having a linear velocity at the pitch circle of not more than 100 feet per minute, and under transmission gears all those running at a higher velocity.

For a pitch t, face b, length of teeth l, and base thickness of tooth h, we have for a tooth pressure P corresponding to a stress S, the general formula:

\[ b \cdot t = \frac{P}{S} \left( \frac{l}{t} \right) \left( \frac{t}{h} \right)^2 \ldots \ldots \ldots (212) \]

and for the proportions of length and thickness already adopted we have:

\[ b \cdot t = 16.5 \frac{P}{S} \ldots \ldots \ldots \ldots \ldots \ldots (213) \]

This assumes that the resistance of the teeth is proportional to their cross section, which is also equally true for those which have the same ratio of b to t to each other, a condition which is often of much service in practice.

§ 226.
PITCH AND FACE OF HOISTING GEARS.

For a hoisting gear of cast iron let:

\[ \left( \frac{P}{R} \right) = \text{the statical moment of the driving force,} \]

\[ Z = \text{the number of teeth,} \]

\[ S = \text{its previously determined pitch radius, in inches,} \]

\[ t = \text{the pitch,} \]

we have for the given dimensions:

\[ t = 0.25 \sqrt{\frac{P}{Z}} \]

\[ t = 0.073 \sqrt{\frac{P}{Z}} \ldots \ldots \ldots (214) \]

\[ t = 0.145 \sqrt{\frac{P}{R}} \]

\[ t = 0.0145 \sqrt{\frac{P}{R}} \ldots \ldots \ldots \ldots (215) \]

the face \( b \) made being

\[ b \cdot z = t \ldots \ldots \ldots \ldots \ldots \ldots (216) \]

These are intended to give a fibre stress \( S \) of about 4200 pounds. The actual stress is properly somewhat less, because the thickness of the tooth at the base is usually more than \( \frac{1}{2} \) \( t \), as assumed in (213).

Since the value of \( \frac{P}{R} \) is the same as the pressure \( P \), we may use (215) in cases in which \( P \) only is given, for rack teeth.

In discussing the preceding formulae, consideration must be given to the elements which are usually given or selected in practice.

Let \( r' \) and \( d' \) be the pitch for two cases respectively, and \( Z' \) and \( S' \) the number of teeth. Also let \( S \) and \( S' \) be the stress at the base of the teeth, and let the constant, \( C \left( \frac{I}{D} \right) \left( \frac{I}{H} \right) \), which in (213) is made equal to 16.8, be called \( C \) or \( C' \); then we have, according to (214):

\[ t = \sqrt{\frac{\pi}{2} \cdot C \cdot t} \quad \frac{S}{S'} \]

whence:

\[ \frac{t}{t} = \sqrt{\frac{C \cdot S}{C' \cdot S'}} \ldots \ldots \ldots (217) \]

and for the radii \( R' \) and \( R'' \):

\[ \frac{R'}{R''} = \sqrt{\frac{C \cdot S}{C' \cdot S'}} \cdot \left( \frac{R}{R''} \right) \ldots \ldots (218) \]

The value of \( C \) depends upon the ratio of the teeth, and upon the value of \( S \) for the material used. If we assume the latter to be the same for both cases, the number of the teeth alone remains to be considered. A reduction in the number of teeth increases the pitch, according to (217); and according to (218) reduces the radius.

\[ \frac{t}{t} = \sqrt{\frac{1}{2}} \cdot \frac{1}{2} \ldots \ldots (219) \]

so that the \( g \) toothed gear will be about \( \frac{1}{2} \) as large as the \( g \) toothed gear, or a \( g \) toothed gear for the same case would be about \( \frac{3}{4} \) as large as a \( g \) toothed gear, and with \( 1.5 \) times greater width of face.

\[ \frac{R'}{R''} = \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2}} \ldots \ldots (220) \]

The constant \( C \), for a given series of gears, should be invariable, and for ordinary spur gears may be taken equal to 16.5, as in (213). For the so-called "thumb teeth," [§ 212], the constant may be much smaller, and hence permit an important reduction in dimensions. The value of \( \frac{h}{t} = \frac{1}{10} \) for wheels of more than ten teeth is not less than 0.7, and introducing this value we get \( C' = 8.4 \), that is 0.5 \( C \), hence "thumb-shaped" profiles are capable of sustaining twice as great a load as the ordinary form.

\[ \frac{t}{t} = \sqrt{\frac{1}{2}} \cdot \frac{1}{2} \ldots \ldots (219) \]

If, however, the teeth are taken in the above ratio of 11 : 7, we would have for the pitch:

\[ \frac{t}{t} = \sqrt{\frac{1}{2}} \cdot \frac{1}{2} \ldots \ldots (219) \]

and the radius \( R = R \cdot \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2}} \ldots \ldots (220) \).

The influence of the stress \( S \) is always important, and it should not be increased above the normal value for the given material, which latter is usually cast iron. An increase of one-fourth in the permissible stress would reduce the pitch and diameter only 7 per cent, but on the other hand it must be remembered that too low a value of \( S \) causes an unnecessary increase in the size and weight, not only of the gears but also of the bearings, frame work and other parts of the machine. The value of \( S \) used above, viz. 4200 pounds, has been shown in practice to give satisfactory results, and there appears to be no good reason for any great variation from it.

When the gears are made of wrought iron, as is sometimes the case, \( S \) may be made much higher, and may indeed be taken double, say 8400 pounds. This gives a reduction in \( t \) in the proportion of \( t \sqrt{0.5} = 0.79 \ t \).
Example 3. For comparison between a wrought iron gear of \( p \) teeth of thumb-shaped outline, with a cast iron gear of \( p \) teeth of ordinary shape, we have:

\[
P' = \sqrt[3]{\frac{0.125 \times 0.125\frac{Z}{S}}{Z'}} = \frac{R}{R'} \frac{0.125}{0.125} = \frac{R}{R'},
\]

and \( P' = 0.75 \).

In Fig. 645 the five cases given in the last three examples are shown on the same scale, side by side. In order to indicate the fact that the moment \( (P'\frac{R}{R}) \) is the same in all cases, the shaft diameter has been shown. It will be apparent that there is no definite relation between the diameter of the shaft and the radius of a gear.

The invariability of the moment, which has been maintained in the preceding examples, does not exist of the tooth pressure \( P' \) upon the driven gear is again transmitted through a second so-called compound gear. If the pinion of a radius \( R \) driving a gear \( R' \), compoud by a pinion \( K_{1} \) on the same shaft into a rack \( R_{2} \), for example, with a given pressure \( P' \), we have from (214):

\[
\frac{P'}{P} = \text{Const.} \frac{K_{1}}{K} \frac{R_{2}}{R_{2}'} \cdot \ldots \ldots \ldots (219)
\]

This gives:

\[
R' = R \sqrt[3]{\frac{K_{1}}{K} \frac{R_{2}}{R_{2}'}} \frac{S}{Z} \frac{Z}{Z'} \ldots \ldots \ldots (220)
\]

But \( R_{2} = Z_{2} \) and \( R'_{2} = Z_{2}' \), and from formula (219):

\[
t' = t \sqrt[3]{\frac{K_{1}}{K} \frac{R_{2}}{R_{2}'}} \frac{S}{Z} \frac{Z}{Z'} \ldots \ldots \ldots (221)
\]

Example 4. A rack with a tooth pressure \( P' \) is driven by a large gear \( R' \), which engages with a \( p \) toothed pinion, in Fig. 646, the teeth being of the usual shape, and the material cast iron.

This is to be replaced by making all parts of wrought iron, and reducing the number of teeth in the rack pinion to \( p \), as shown in Fig. 647, teeth being also altered to the thumb-shaped form. Then have \( C_{2} = 0.45 \), \( X_{2} = 0.35 \), and hence:

\[
P' = \frac{K_{2}}{K} = \frac{S}{Z} = \frac{Z_{1}}{Z_{1}'} \sqrt[3]{C_{2}} \frac{S}{Z} \frac{Z}{Z'} \ldots \ldots \ldots (226)
\]

Hence we get:

\[
t' = t \sqrt[3]{\frac{C_{2}}{C} \frac{S}{Z} \frac{Z}{Z'} \ldots \ldots \ldots \ldots (227)}
\]

By selecting the number of teeth we may make:

\[
\frac{Z_{1}'}{Z_{1}} = \frac{Z}{Z'}
\]

and then obtain:

\[
t' = t \sqrt[3]{\frac{C_{2}}{C} \frac{S}{Z} \frac{Z}{Z'}} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \{}
dynamic action of shock and vibration also increases. For cast iron we may take:

$$ S = \frac{9,600,000}{v + 2164}$$

in which $v$ is the linear velocity in feet per minute. For steel, $S$ may be taken 2½ times, and for wood, ½ the value thus obtained. For cast iron we obtain, for:

<table>
<thead>
<tr>
<th>Cast Iron</th>
<th>200</th>
<th>500</th>
<th>1000</th>
<th>2000</th>
<th>3000</th>
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<td>$S = 5420$</td>
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<td>$3473$</td>
<td>$3258$</td>
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<td>$2812$</td>
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</table>

And for wood:

$$ S = 2574, 2524, 2456, 2285, 1935, 1820, 1757, 1681, 1540 $$

The velocity $v$ may be obtained when $u$ and $R$ (the latter in inches) are given, by the following formula:

$$ v = \frac{R \cdot u}{12} = 0.0339 \cdot R \cdot u $$

The selection of a proper value for $v$ will be discussed below. It is also found that the breadth of face $b$ should increase with the increase of $P$. Tredgold states that the pressure per inch of face, that is $P$, should not exceed 400 pounds. This, however, is not to be followed implicitly, since pressures as high as 100 times have been successfully used in practice. It is better, however, to consider the question of wear from the product of $P$ into $b$, which should not exceed a predetermined maximum.

It is found that if $P \times b$ exceeds 67,000 the wear becomes excessive. In a pair of wheels where the teeth of both are of iron, the greatest wear comes upon the teeth of the smaller wheel. In this case we may make

$$ \frac{P}{b} = \text{not more than 28,000} $$

and if possible it should be taken at less than this value. For smaller forces this constant, which we may call the co-efficient of wear and designate as $A$, may readily be made as low as 12,000, and even 6,000, without obtaining inconvenient dimensions. When the teeth are of wood and iron the wear upon the iron may be neglected, as the wear comes almost entirely upon the wooden teeth. For wooden teeth the value of $A$ should not exceed 28,000, and is better made about 15,000 to 20,000. It is impossible to give exact values in such constructions, and it must be left to the judgment of the designer as to how far it may be advisable to depart from the values obtained from existing examples.

It must be remembered that the different values of $A$ do not appreciably affect the strength, but rather control the rapidity of wear. When sufficient space is available a low value can be given to the co-efficient of wear, it is advisable to do so; if this cannot be done, the co-efficient which is selected will give an indication of the proportional amount of wear which may be expected.

In cases where a number of wheels gear into one another, it is better to take, instead of the number of revolutions of the common wheel, the number of tooth contacts, that is the product of the revolutions and the number of wheels in the group. If $R$ is given, as in the case of an engine with water wheels, fly-wheels, &c, $P$ is also known, and since $A$ can be chosen we have, taking $N$ to be the horse power transmitted:

$$ b = \frac{P}{A} = \frac{63,000}{91,000} $$

hence from (213) for ordinary teeth, $t = \frac{168}{S \cdot b} = \frac{168}{A}$

and for thumb shaped teeth

$$ \nu = \frac{168}{S \cdot b} = \frac{8.4}{A} $$

If, however, as occurs in many cases, $R$ is not previously de-

termined, the choice of the number of teeth $Z$ is unrestricted. In such cases we have for the width of face $b$:

$$ b = \frac{36,000}{A} $$

If we give to $A$ the successive values 30,000, 25,000, 20,000, 15,000, and 10,000 we get the following numerical relations:

<table>
<thead>
<tr>
<th>Common Teeth</th>
<th>Common Teeth</th>
<th>Thumb Teeth</th>
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<tr>
<td>$b = \frac{P}{30,000}$</td>
<td>$b = \frac{P}{25,000}$</td>
<td>$b = \frac{P}{20,000}$</td>
</tr>
<tr>
<td>$N = 13.44$</td>
<td>$N = 15.44$</td>
<td>$N = 19.83$</td>
</tr>
<tr>
<td>$Z = b \cdot t$</td>
<td>$Z = b \cdot t$</td>
<td>$Z = b \cdot t$</td>
</tr>
<tr>
<td>$\frac{t}{y} = \frac{500}{n \cdot s}$</td>
<td>$\frac{t}{y} = \frac{400}{n \cdot s}$</td>
<td>$\frac{t}{y} = \frac{300}{n \cdot s}$</td>
</tr>
<tr>
<td>$\frac{y}{s} = \frac{212}{n \cdot s}$</td>
<td>$\frac{y}{s} = \frac{210}{n \cdot s}$</td>
<td>$\frac{y}{s} = \frac{168}{n \cdot s}$</td>
</tr>
<tr>
<td>$\frac{y}{s} = \frac{252}{n \cdot s}$</td>
<td>$\frac{y}{s} = \frac{220}{n \cdot s}$</td>
<td>$\frac{y}{s} = \frac{168}{n \cdot s}$</td>
</tr>
<tr>
<td>$\frac{y}{s} = \frac{42}{n \cdot s}$</td>
<td>$\frac{y}{s} = \frac{36}{n \cdot s}$</td>
<td>$\frac{y}{s} = \frac{20}{n \cdot s}$</td>
</tr>
</tbody>
</table>

For transmission gears the minimum number of teeth should not be fewer than 20, in order that the unavoidable errors of construction shall not cause excessive wear; for quick-running gears it is desirable to have still more teeth. The gear wheels on high speed turbines seldom have fewer than 40, and often as many as 80 teeth. When wood and iron teeth are used, the least wear is produced when the wooden teeth are on the driver, because the action begins at the base of the teeth and passes toward the point, while on the driven gear the action is reversed. If desired a number of teeth $Z$ can be calculated which will give a desired ratio $b : t$. To combine formulæ (223) and (226) we obtain the useful relation:

$$ Z = \frac{36,000}{A} \cdot \frac{n \cdot s \cdot N}{16,800} $$

This shows the important influence of $A$ upon $Z$, and the effect of the number of teeth upon the wear, also the important relation of the tooth profile, since the constant 16.8 (or for thumb teeth 8.4) appears in the second power. It is also seen that $Z$ is dependent on the square of $u$, and the square of $N$, other things being constant. These points indicate the methods of obtaining the least stress.

The value of $b : t$ is sometimes made as great as 5. For wider faces and sometimes for narrower, the rim of the gear is made of two adjoining parts.

Example 1.—A water wheel of horse power, 26 feet, 3 inches in diameter, moving with a velocity at the circumference of 225 feet per minute, is to be provided with an internal gear wheel, the pitch circle being at inches less radius than that of the water wheel, and gearing into a pinion which is to make 40 revolutions per minute. We have:

$$ a = 225 $$

and $b = 10$; also $\sqrt{b} = 10$; $\sqrt{b} = 10$. This gives a permissible stress $S = 400$ lbs. nearly. We will choose for the smaller wheel $P = 15,000$, which gives $b = \frac{P}{A} = 25,000$. Hence $b = 625$, and $\frac{t}{y} = 13\frac{1}{2}$. We then have from (227) $= \frac{420,000}{13\frac{1}{2}} = \frac{420,000}{375}$. We then have $= \frac{500}{S \cdot b} = \frac{500}{A}$. For we make $= \frac{30,000}{S \cdot b}$, the wheel may be divided into 15 segments of 15 teeth each. For the driven wheel we have $Z = 50, Z = \frac{40}{348} = 27\frac{1}{2}$, whence $R = 27\frac{1}{2}$. For $= \frac{27\frac{1}{2}}{3}$. $= \frac{27\frac{1}{2}}{3}$. $= \frac{27\frac{1}{2}}{3}$.
THE CONSTRUCTOR.

Example 1—In a given train of gearing, Fig. 46, in which the corresponding teeth of both pairs are of the same size, the force transmitted in each case is inversely as the number of revolutions. In order to have the coefficient of wear alike in both cases it is only necessary to make all the gears of the same face. An example of this kind may be found in the back gearing of many lathes.

Example 2—Let it be required to construct a pair of durable gears of wooden and iron teeth under the following conditions: $N = 5$, $n = 8$, $m = 60$, and $f = a$. We may make $a = 500$, which gives, from (99), $S = 350$, and as great durability is required we will take $A = 350$. The values in (118) give:

$$P = \frac{356,000}{15.8 \times 9,200^2} \times \frac{5 \times 5 \times 5}{2} = 8.8$$

which we may call teeth. We have from (97)

$$f = 150$$

$$S = 460$$

and $b = \frac{356,000}{15.8 \times 9,200^2} = 2.33$.

If we take the pitch-shaped teeth, we may make $b = 2.5$. If we make $f = 200$, $S = 350$ and we take $A = 350$.

This gives for the driver:

$$P = \frac{356,000}{15.8 \times 13,000^2} \times \frac{5 \times 5 \times 5}{2} = 4.7$$

say 45 teeth, and $S = \frac{2}{3}$. We get $f = 200$, $S = 350$, $A = 350$.

If we take the thumb-shaped teeth, and make $b = 2.5$, we get:

$$P = \frac{356,000}{15.8 \times 13,000^2} \times \frac{5 \times 5 \times 5}{2} = 11$$

say 120, and $S = \frac{2}{3}$. We get $f = 200$, $S = 350$, $A = 350$.

This gives smaller teeth, but larger radii than when the common form is used.

When steel is used for gear wheels, special proportions are obtained. It is not too much to say that the value of the coefficient of wear $A$ should be taken twice as great as for cast iron. The gears, however, may be taken 4 or 5 times too permissible for cast iron. Taking these points into consideration in formula (118) we see that $A$ would reduce the number of teeth by $1/2$, and $S$ would increase it by $1/3$, that is, about 20% so that the net increase would be $11$.

If the above values are accepted, it may therefore be laid down as a rule that steel teeth should have more teeth for the same service than cast iron gears. The ratio of pitch to pin may be made quite large, and in the case of double spiral gears (as in Fig. 46) the ratio $ \frac{f}{S} $ is sometimes made as great as 7 or 8.

If the formula for thumb teeth be used, instead of the usual shape, the constant $a$ will give satisfactory results. The value obtained for the pitch is that for the normal pitch $\pi / 48$, but the width of face is the actual width, $4 \pi$, in Fig. 65, $F = 4 \pi$. In Fig. 58.

Example 3—Suppose the wheels given in Example 2 to be made with double spiral teeth of steel. We take $A = 58,000$, and $b = \frac{2}{3}$ also $S = 12,500, f = \frac{1}{3}$, $A = 58,000$.

We then get:

$$P = \frac{356,000}{15.8 \times 7,300^2} \times \frac{5 \times 5 \times 5}{3}$$

We have $f = 240$, $S = 12,500$, $A = 58,000$.

also $b = \frac{356,000}{15.8 \times 7,300^2} = 4.4$.

If we take $S = 12,500$, we get $P = 140$ and $b = 140$. If $f = 60$, we have

$$f = \frac{240}{60} = 4$$

and

$$b = \frac{2}{3}$$

We may take $f = 60$, which gives $a = 64$, $b = 0.572$ and

$$b = \frac{2}{3}$$

We have then finally $A = 11.4$, $S = 10.47$. 220.

EXAMPLES AND ACTUAL CASES

The following examples taken from actual practice will be of interest (see Table on following page).

No. 1. From the driving gear of the main steam engine of Fleming's Spinning and Weaving Mill in Bombay. The toothed fly-wheel is the driver, and the teeth are shrunk over, as shown in Fig. 46. The coefficient of wear for the driven gear seems high, and does not indicate long endurance.

No. 2. A toothed fly-wheel engaging with a pair of equal spur gears; 300 horse-power transmitted by each gear, making a total of 600 horse-power. The value for $P / b$ must therefore be multiplied by 2; see last column of the table.

No. 3. This is from the air compressor for the atmospheric railway of St. Germain (now abandoned). $P / b$ is evidently too high, as would probably have become apparent had the gears continued in operation.

No. 4. $P / b$ is very high, but the small number of revolutions keeps the value of $P / b$ within reasonable limits.

No. 5 and 6. These are from the great water wheel at Greenock. The pressure at the rim is great, but the teeth have worn well in practice, as might have been predicted from the moderate values of $P / b$. The value of the latter is almost the same for No. 6, as for No. 5, hence the wear should be about the same for both gears.

No. 7. The teeth in the smaller gear are thinner than those of the large fly-wheel, hence the two values for $S$. Probably the larger wheel was originally made with wooden teeth.

No. 9. Notwithstanding the high pressure the value of $P / b$ is reasonably small. The stress upon the teeth is quite high, as is also the case with No. 4, and lower stresses are to be recommended.

No. 10. This is one of the most noteworthy examples of the whole collection, on account of the very slight wear exhibited. The wooden teeth on the large wheel, (the fly-wheel of the steam engine of the Kelvinville Paper Mill at Glasgow) ran for 265 years, for 20 hours per day, with a wear upon the teeth, measured at the pitch circle, of only about 3/16 inch. For the first half of this time the engine indicated 84 horse-power, at 38 revolutions. The teeth were lubricated twice a week with coal tar and graphite. The long endurance is doubtless partially due to the great care which the teeth received, they having been cut upon the wheel in place, but also to the moderate co-efficient of wear.

No. 11. The teeth were found too small in practice, as is indicated by the stress of 3500 pounds; from formula (222) we obtain $S = 377$ pounds.

No. 12. Two gears with wooden teeth engage with a single pinion on the screw propeller shaft. The teeth are in two sets of 4/10" width of face each.

No. 13. Very high pressure, which must appear in the teeth upon the spider; therefore, it is necessary to be kept in good condition, owing to the high value of $P / b$.

No. 15. These teeth appear weak, as has been shown by repeated breakages. The wear must be rapid, as indicated by the high value of $P / b$.

No. 17. These gears, designed by Fairbairn, were intended ultimately to transmit double the power at first given, in which case the stress would reach over 4000 pounds, which is admissible but the value of $P / b$ would then become rather too high to indicate very great endurance.

No. 20. The value of $P / b$ seems too high for the wooden teeth, it is almost too great also for the iron teeth, and it must be remembered that with wooden and iron teeth, the wear is almost entirely upon the wooden teeth.

No. 22. These gears are from an establishment which has used hypbolodial gears with much success for power transmission. The angle of the axis is 90°. The use of wooden teeth upon the driver is to be criticized, as tending to increase the liability to wear.

THE DIMENSIONS OF GEAR WHEELS.

§ 230.

THE RIM.

The ring of metal upon which the teeth of a gear wheel are placed is called the rim. For cast iron spur gears, the thickness of the rim is given by the formula:

$$\delta = 0.4 f + 0.25'$$
### EXAMPLES OF TRANSMISSION GEARING.

<table>
<thead>
<tr>
<th>No.</th>
<th>N</th>
<th>a</th>
<th>R</th>
<th>Z</th>
<th>t</th>
<th>b</th>
<th>P</th>
<th>S</th>
<th>S</th>
<th>P</th>
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### BEVEL GEARS.

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<td>6.3</td>
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<td>2860</td>
</tr>
<tr>
<td>21</td>
<td>50</td>
<td>93</td>
<td>218</td>
<td>25.6</td>
<td>10.8</td>
<td>75</td>
<td>32</td>
<td>2.1</td>
<td>6.3</td>
<td>1236</td>
<td>1313</td>
</tr>
</tbody>
</table>

### HYPERBOLOIDAL GEARS.

<table>
<thead>
<tr>
<th>No.</th>
<th>N</th>
<th>a</th>
<th>R</th>
<th>Z</th>
<th>t</th>
<th>b</th>
<th>P</th>
<th>S</th>
<th>P</th>
<th>P</th>
<th>REMARKS</th>
</tr>
</thead>
</table>
See Fig. 647. The rim is thickened in the middle or at one edge to \( \frac{6}{5} \) \( \delta \), and also stiffened by a rib, and for gears of fine pitch the section of the rim is curved, which harmonizes well with arms of oval section. According to (222) a pitch of \( \frac{1}{4} \) would give a rim thickness \( \delta = \frac{0.014}{5} + \frac{0.125}{5} = \frac{0.0625}{5} \) or a little over \( \frac{1}{5} \), and for a pitch of \( \frac{1}{8} \), \( \delta = \frac{0.375}{7} \).

For bevel gears of cast iron the rim is made \( \frac{6}{5} \) \( \delta \) thick at the outer edge, and of the various forms shown in Fig. 648.

For wooden teeth it is necessary to have a deeper and stronger rim, the dimensions being dependent somewhat upon the method of inserting the teeth. The proportions for spur gears are shown in Fig. 649, and for bevel gears in Fig. 650. For very wide faces the wooden teeth are made in two pieces and a stay bar cast in the mortise.

Small pinions are often cast solid, and when subjected to heavy pressures are strengthened by shrouding, as shown in Fig. 651, and sometimes this shrouding is turned down to the pitch line.

For double spiral gears of steel (see § 222) shrouding is to be recommended, and is very generally used. The use of shrouding especially assists in securing good steel castings, for the great shrinkage of the steel, nearly two per cent, tends to produce warped and twisted castings.

Small pinions are sometimes cut from solid wrought iron, in which case the shrouding must be omitted.

\[ \begin{align*}
\text{Table of Gear Wheel Arms.} \\
\hline \\
\text{Value of} & \begin{array}{c|c|c|c|c|c|c|c}
\frac{\beta}{\delta} \\
\hline
\theta & \frac{1}{6} & \frac{1}{12} & \frac{1}{18} & \frac{1}{24} & \frac{1}{30} & \frac{1}{36} & \frac{1}{40} \\
\hline
1.50 & 0.30 & 0.33 & 0.37 & 0.40 & 0.43 & 0.46 & 0.50 \\
1.75 & 0.41 & 0.46 & 0.51 & 0.56 & 0.61 & 0.66 & 0.70 \\
2.00 & 0.52 & 0.58 & 0.64 & 0.70 & 0.76 & 0.82 & 0.88 \\
2.25 & 0.62 & 0.70 & 0.78 & 0.85 & 0.93 & 1.01 & 1.08 \\
2.50 & 0.73 & 0.82 & 0.92 & 1.01 & 1.11 & 1.21 & 1.30 \\
2.75 & 0.84 & 0.94 & 1.05 & 1.15 & 1.26 & 1.37 & 1.48 \\
3.00 & 0.95 & 1.06 & 1.17 & 1.28 & 1.39 & 1.50 & 1.62 \\
\hline
\end{array} \\
\end{align*} \]