APPENDIX.

I. The Axis of Instantaneous Rotation.

When a rigid body is in motion, it is turning, during every separate instant, about some straight line or other. (v. page 17.)

For an analytical proof of the existence of this line see Whewell's Dynamics. (Art. 120.)

The following is extracted, by permission of the Author, from Earnshaw's Statics. (Art. 109.)

"Let $P$, $Q$, (fig. 18.) be any two particles of a rigid body; $PP'$, $QQ'$, the paths which they describe during the same instant; $A$, $B$, the centres of curvatures of these paths; then the line joining $A$, $B$, will be the axis about which the whole body turns during this instant."

"For the lines which join the successive contempoaneous positions of $P$ and $Q$, while they are respectively passing to $P'$ and $Q'$, will form a species of conical surface; and since, by reason of the rigidity of the body, they are all of the same length, the planes $PAP'$, $QBQ'$, in which the curves $(PP', QQ')$ formed by their extremities lie, must be parallel. Now since $P$ describes round $A$ the angle $PAP'$ in the same time that $Q$ describes round $B$ the angle $QBQ'$, the trapezium $PABQ$ turns in the same time round $AB$ and comes into the position $P'ABQ'$ (for $AP = AP'$, and $BQ = BQ'$, since $A$, $B$, are the centres of curvature of $PP'$, $QQ'$). Con-
sequently the motion of every particle of the body situated in \( PQ \) takes place about the axis \( AB \).”

“Hence the motions of the points \( P \) and \( Q \) take place about \( AB \), and therefore \( AB \) must be perpendicular to \( PAP' \), \( QBQ' \), the planes of these motions. In like manner if the motion of any other particle \( R \) take place about a point \( C \), \( AC \) must be perpendicular to the planes of motion \( PAP' \), \( RCR' \); hence both \( AB \) and \( AC \) are perpendicular to \( PAP' \), which is impossible, (Euc. xi. 13.) unless they coincide; in which case \( C \) is a point in \( AB \), and the motion of \( R \) takes place about \( AB \); and since \( R \) is any particle, the motion of every particle takes place about \( AB \), that is, the whole body turns during the instant about the straight line \( AB \).”

If any point in the body be fixed the axis must pass through this point.

For let \( O \) be the point and join \( AO \); then we may consider \( PAO \) as a crooked but rigid rod moveable about the fixed point \( A \), and it is manifest that while one extremity \( P \) moves through \( PP' \), the other cannot remain at rest unless it lies in \( AB \).

In this case the motion of any one point \( P \) determines the motion of every other point.

For if \( PO \) be joined, the rod \( PO \) considered as a rigid body must be turning during every separate instant about some axis passing through \( O \); and it is shewn in the course of the above demonstration that the plane of the motion of any other point \( Q \) in the rigid body is parallel to the plane of the motion of any point in \( OP \). Therefore the motion of \( Q \) is about the same axis as that of \( OP \); and it is clear from note (1) that the angular velocities are the same.

If the body is perfectly free, it has during every instant a simple rotatory motion about some axis pass-
ing through the centre of gravity; except in the case when all the particles move in equal and parallel straight lines, that is, when the body has a mere motion of translation.

To prove this it is necessary to establish the following Dynamical property of the centre of gravity.

If a motion of translation be communicated to a body which has a simple rotatory motion about any axis passing through its centre of gravity, the motions will subsist together, and each will continue to affect the body precisely as it would have done if the other had never existed.

Suppose a velocity \( v \) in the direction of a line which makes angles \( \alpha, \beta, \gamma \), with the co-ordinate axes to be communicated to every particle of the rigid system in note (11).

Then the resolved parts of the effective forces which act on a particle \( M \) situated at the point \( P \) are

\[
-m \omega y + m v \cos \alpha, \text{ in the direction } Gx,
\]

\[
m(\omega x + v \cos \beta) \quad \ldots \ldots \ldots \quad Gy,
\]

\[
m v \cos \gamma \quad \ldots \ldots \ldots \quad Gz.
\]

And the resolved parts of all the elementary effective forces are reducible to

(i.) Three forces applied at \( G \); viz.

\[
-\omega \Sigma (my) + v \cos \alpha \Sigma (m),
\]

which by the property of the centre of gravity (if \( \mu = \Sigma (m) \) the mass of the system)

\[
= \mu v \cos \alpha, \text{ in the direction } Gx,
\]

\[
\omega \Sigma (mx) + v \cos \beta \Sigma (m) = \mu v \cos \beta \quad \ldots \ldots \ldots \quad Gy.
\]

\[
v \cos \gamma \Sigma (m) = \mu v \cos \gamma \quad \ldots \ldots \ldots \quad Gz.
\]
(ii.) Two couples; viz.

\[ \Sigma m z (-\omega y + v \cos \alpha) - \Sigma m x v \cos \gamma \]

\[ = - \omega \Sigma m y z \text{ in the plane } z x, \]

\[ \Sigma m z (\omega x + v \cos \beta) - \Sigma m y v \cos \gamma \]

\[ = \omega \Sigma m x z \text{ in the plane } z y. \]

(iii.) Two couples; viz:

\[ \Sigma m x (\omega x + v \cos \beta) \text{ and } \Sigma m y (\omega y - v \cos \alpha), \]

which together = \( \omega \Sigma m (x^2 + y^2) \) in the plane \( xy \).

The whole elementary effective forces are therefore equivalent to a force equal to the resultant of the forces (i) = \( \sqrt{\mu^2 v^2 (\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma)} \) = \( \mu v \) applied at the centre of gravity, in a direction making angles \( \alpha, \beta, \gamma \), with the axes, and to the resultant of the two sets of couples (ii) and (iii); and it is manifest that if the rotatory motion were suppressed, the resultant force would still be \( \mu v \), and the resultant couple would vanish; and therefore conversely that the consequence of applying a force = \( \mu v \) at the centre of gravity of the body is to give every particle of it a mere motion of translation with a velocity \( v \), in the direction of the force, whether the body has or has not a rotatory motion; and that on the other hand the couples (ii) and (iii), which are necessary and sufficient to produce a simple rotatory motion about an axis through \( G \), remain the same, whatever be the value of \( v \), and therefore of \( \mu v \).

Suppose now that \( AB \) (fig. 19.) the line containing the centres of curvature of the paths of every particle of the body in absolute space, does not pass through the centre of gravity \( G \) of the body, and that the angular velocity of the body about it is \( \omega \). Then if about \( Gg \) a line through \( G \) parallel to \( AB \), we suppose two angular
velocities each = \omega to be communicated to the body in opposite directions, we shall have a simple rotatory motion with an angular velocity \omega about \(Gg\) and a motion of translation perpendicular to the plane \(GgAB\); and since \(Gg\) passes through the centre of gravity these motions are independent, that is, the body has during the instant, a simple rotatory motion about an axis through \(G\).

Hence also it is impossible for a free rigid body to have a simple rotatory motion about any axis which does not pass through the centre of gravity.

For the rotatory motion which it has for an instant about any such axis can always be resolved into a simple one about an axis through the centre and a motion of translation. Hence the axis of the screw spoken of in pp. 24, 25, always passes through the centre of gravity of the body.

We may also illustrate the above principles by a reference to the motions of the Earth and Moon in absolute space.

If the Earth had no rotatory motion about its own axis we might naturally consider \(SK\) (fig. 20.) perpendicular to the plane of the ecliptic as the instantaneous axis (\(AB\) figs. 18 and 19.). The motion about it would be equivalent at every instant to a motion of translation in the direction of a tangent to the Earth's orbit, and a simple rotatory motion about an axis through the centre of gravity of the Earth parallel to \(SK\). In the course of a year the Earth would turn once round on this axis in the direction \(WLE\), and to a spectator on the Earth's surface the Sun would appear to describe in the same direction a circle round the centre of gravity of the Earth in the plane of the ecliptic, and the fixed stars parallel circles in the opposite direction. And this motion would be totally independent of the motion
of translation, of which the spectator would only be aware from the consideration that if it were suppressed the Sun's apparent motion would be in the opposite direction.

But no such apparent annual motion of the fixed stars is observed. We may therefore conclude that the Earth has no such simple rotatory motion, and that its motion round the Sun is a simple motion of translation of the centre of gravity; which would also appear from Dynamical considerations. It is observed that the Moon presents always the same face towards the Earth. Hence all the particles in it describe similar curves about a line through the centre of gravity of the Earth perpendicular to the Moon's orbit. This axis is therefore always an instantaneous axis of the Moon. Hence the Moon has a simple rotatory motion in the same direction with its motion of translation about an axis through its centre of gravity parallel to the instantaneous axis, and turns once round on this axis during a revolution of its centre of gravity round that of the Earth.

If now we suppose one or more rotatory motions to be communicated about other axes through the centre of gravity, the motion of translation at every instant will not be affected, and the simple rotatory motion about the centre of gravity at every instant, upon which the appearances of the heavens depend, will be compounded of these.

II. Principal Axes and Moments of Inertia.

Through every point in a material system at least three straight lines may be drawn, in directions mutually at right angles, for which, when they are severally taken as the axis of $x$, each of the quantities $\Sigma (m x z)$, $\Sigma (m y z)$ vanishes.
In a rigid body or system these lines are called principal axes. (v. page 28.)

If $GP$ (fig. 11.) make angles $\alpha$, $\beta$, $\gamma$, with the co-ordinate axes, and the co-ordinates of $Q$, the position of any particle $m$, be $x$, $y$, $z$; as in note (13).

Moment round $GP$ ($P$) = $\Sigma m \ (QR)^2$

$$= \Sigma m \ (x^2 + y^2 + z^2 - GR^2)$$

$$= \Sigma m x^2 + \Sigma m y^2 + \Sigma m z^2$$

$$- \Sigma m x^2 \cos^2 \alpha - \Sigma m y^2 \cos^2 \beta - \Sigma m z^2 \cos^2 \gamma$$

$$- 2 \Sigma m x y \cos \alpha \cos \beta - 2 \Sigma m x z \cos \alpha \cos \gamma$$

$$- 2 \Sigma m y z \cos \beta \cos \gamma.$$

Let $A = \Sigma m y^2 + \Sigma m z^2$, which is manifestly the moment round $Gz$,

$A' = \Sigma m y z$, and so on for the other axes.

Then $P = A \cos^2 \alpha + B \cos^2 \beta + C \cos^2 \gamma$

$$- A' \cos \beta \cos \gamma - B' \cos \alpha \cos \gamma - C' \cos \alpha \cos \beta.$$

And if we take any point $x, y, z$ in $GP$ at a distance $p$ from $G$, we shall have

$$pp' = A x^2 + B y^2 + C z^2 - A' y z - B' x z - C' x y.$$

Now the value of $P$ evidently depends upon the values of $\alpha$, $\beta$, $\gamma$, that is upon the position of $GP$; and $p$ being arbitrary we may take it equal to any function of the same quantities;

let it = \frac{1}{\sqrt{n} \cdot P};

\therefore \quad pp' = \frac{1}{n}.
and \( Ax^2 + By^2 + Cz^2 - A'yz - B'xz - C'xy = \frac{1}{n} = c \),

is the equation to the locus of the extremity of \( p \). We see at once that it is a surface of the second order whose centre is \( G \); and since \( P \) can never vanish, except in the particular case when all the particles lie in the straight line \( GP \), \( p \) is always finite, and the surface is an ellipsoid.

If now the axes of the co-ordinates be transformed so as to coincide with the Geometrical Axes of the ellipsoid, the form of the equation becomes

\[
Ax^2 + By^2 + Cz^2 = c;
\]

For these axes therefore the quantities \( A', B', C' \), respectively \( = 0 \); and therefore each of them is a principal axis of the body or system.

Now a geometrical axis of an ellipsoid coincides with the normal at the point where it meets the surface; at this point therefore we have

\[
\frac{Ax - C'y - B'z}{x} = \frac{By - A'z - C'x}{y} = \frac{Cz - B'x - A'y}{z},
\]

or

\[
\frac{A \cos \alpha - C' \cos \beta - B' \cos \gamma}{\cos \alpha} = \frac{B \cos \beta - A' \cos \gamma - C' \cos \alpha}{\cos \beta} = \frac{C \cos \gamma - B' \cos \alpha - A' \cos \beta}{\cos \gamma} = P;
\]

whence we obtain

\[
(P - A) \cos \alpha + C' \cos \beta + B' \cos \gamma = 0,
\]

\[
C' \cos \alpha + (P - B) \cos \beta + A' \cos \gamma = 0,
\]

\[
B' \cos \alpha + A' \cos \beta + (P - C) \cos \gamma = 0.
\]
and by elimination,

\[(P-A)(P-B)(P-C) - A'^2(P-A) - B'^2(P-B) - C'^2(P-C) + 2 \cdot A'B'C' = 0,\]

in which the roots of \(P\) are of course the moments of inertia about the geometrical axes of the ellipsoid.

To shew that the roots of this equation are real, assume \(A' = 0, B' = 0\); then the factor \(P - C\) disappears and the equation becomes a quadratic, and if this has for its roots \(P_1, P_2\), which we find to be possible quantities, and we substitute successively in the original equation

\[-\infty, P_1, P_2, \text{ and } +\infty,\]

we obtain results alternately positive and negative, whence we conclude that there are three real roots between these limits, (vide Cauchy, Exercices, Vol. III. p. 5.)

The above ellipsoid is manifestly the same with that which Poinsot calls the central ellipsoid.

The moments of inertia about the principal axes passing through the centre of gravity are called the principal moments of inertia of the body.

The moment about the major axis of the ellipsoid is clearly less, and that about the minor axis greater, than that about any other axis whatever.

If two of these are equal the ellipsoid whose equation we have found above becomes a spheroid, and every diameter in the equatorial plane is a principal axis, or the number of principal axes is infinite.

If the three principal moments are equal, the ellipsoid becomes a sphere, and every diameter is a principal axis.
When the co-ordinate axes are the principal axes through the centre of gravity, the value of $P$ is reduced to

$$A \cos^2 \alpha + B \cos^2 \beta + C \cos^2 \gamma.$$ 

From this we can deduce the moment about any axis which does not pass through the centre.

For if a rigid body or system revolve about a fixed axis $AB$ (fig. 19.) with an angular velocity $\omega$, the resultant effective couple in the plane $Pag$ perpendicular to $AB = \omega \times (\text{moment of inertia } (A) \text{ round } AB)$.

But if the axis were not fixed the motion might be resolved as before into a rotatory motion round $Gg$, and a motion of translation; the former of which gives us a couple $= \omega \times (\text{moment } (G) \text{ round } Gg)$, and the latter a force applied at $G$

$$= \mu \text{ (velocity of translation)}$$
$$= \mu (\omega \cdot a) \text{ if } GO = a, \text{ see note (5)},$$

which is equivalent to a force $\mu \omega a$ applied at $O$ and a couple $= \mu \omega a \cdot a$ in a plane perpendicular to $GOA$.

And the force is destroyed by the resistance of the fixed axis;

$$\therefore \omega A = \omega G + \omega \mu a^2,$$
$$A = G + \mu a^2.$$

The principal moments of a homogeneous solid body are readily determined by integration.

For if $\mu = \Sigma (m)$ be a continuous function of $x$ and $y$, the value of an individual elementary portion $\Delta \mu = m$ of it must entirely depend on the values of $x$ and $y$ at the point where that portion is situated.
Therefore the corresponding elementary portion $\Delta C$ of the moment, which $= \Delta \mu \cdot (x^2 + y^2)$, must depend entirely on the values of $x$ and $y$, and must therefore be a function of $\mu$.

But $\frac{\Delta C}{\Delta \mu} = x^2 + y^2$;

$\therefore$ taking the limits, $d_\mu C = x^2 + y^2$.

Similarly $d_\mu A = y^2 + x^2$,

$d_\mu B = x^2 + z^2$.

If the density of the body vary according to a given law of the position of a particle,

$$d_x d_y d_z \mu = \rho \times f(x, y, z);$$

$\therefore$ $d_z \mu = \rho \int_x \int_y f(x, y, z),$

and $d_z C = \rho \int_x \int_y (x^2 + y^2) \times f(x, y, z)$

and similarly for $A$ and $B$.

If the body is homogeneous,

$$d_x d_y d_z \mu = \rho; \therefore d_z \mu = \rho \int_x \int_y 1,$$

$\therefore$ $d_z C = \rho \int_x \int_y (x^2 + y^2) = \rho \int_x \int_y x^2 + \rho \int_x \int_y y^2$,

and similarly for $A$ and $B$.

For a plane surface $z = 0$; and $d_x d_y \mu = \rho$;

$\therefore C = \rho \int_x \int_y x^2 + \rho \int_x \int_y y^2$,

$$A = \rho \int_x \int_y y^2, \quad B = \rho \int_x \int_y x^2;$$

$\therefore C = A + B.$
If $AB$ (fig. 21.) be an uniform physical line whose middle point is $G$, which is clearly the centre of gravity, and a line $Gg$ perpendicular to it be taken for the axis of $(x)$, we shall evidently have $z = 0$ for every point in this line:

$$\therefore \Sigma (max) = 0, \quad \Sigma (myz) = 0,$$

and every line perpendicular to $AB$ is a principal axis thereof.

To find the principal moment, take $GA$ for the axis of $x$, then $y = 0$ and $d_x \mu = \rho$;

$$\therefore C = \rho \int x^2$$

$$= \rho \frac{x^3}{3} + C,$$

which from $x = -\frac{a}{2}$ to $x = +\frac{a}{2}$

$$= \rho \cdot \frac{1}{3} \cdot \frac{a^3}{4}$$

$$= \mu \cdot \frac{a^2}{12}.$$

The moment about an axis through $A$ parallel to $Gg$

$$= \mu \cdot \frac{a^2}{12} + \mu \cdot \frac{a^2}{4}$$

$$= \mu \cdot \frac{a^2}{3}.$$

The moment about an axis $Gh$ inclined at an angle $(a)$ to $Gg$

$$= \mu \cdot \frac{a^2}{12} \cos^2 a.$$
If $ABCD$ (fig. 22.) be a rectangular parallelogram, any axis perpendicular to its plane will be a principal axis; and lines through $G$ parallel to the sides will evidently be principal axes at $G$, since for every product $+ m x y$ we shall have a corresponding product $- m x y$, and $\Sigma m x y = 0$, and if $AD = a$, $AB = b$, $GN = y$,

$$d_y \mu = \rho \cdot MM' = \rho \cdot AD;$$

\[ \therefore \text{moment round } G x (A) = \int y^2 \cdot \rho a \]

\[ = \rho a \frac{b^2}{12} = \mu \frac{b^2}{12}, \]

\[ B = \rho b \frac{a^2}{12} = \mu \frac{a^2}{12}. \]

Hence in this case

\[ C = \mu \cdot \frac{a^2 + b^2}{12}, \]

and if we suppose this axis to pass through the centre of gravity of any number of rectangles of the same size and density, and exactly parallel to $ABCD$, the moment of the system about this axis will evidently be obtained by multiplying $C$ by the number of these planes. Hence the principal moments of a rectangular parallelepiped whose base is $ABCD$ will be

\[ C = \rho abc \cdot \frac{a^2 + b^2}{12} = \mu \frac{a^2 + b^2}{12}, \quad \text{and similarly}, \]

\[ B = \mu \cdot \frac{a^2 + c^2}{12}, \quad A = \mu \cdot \frac{b^2 + c^2}{12}. \]
Whenever a homogeneous solid body can be divided by three planes, passing through the centre of gravity at right angles to each other, into perfectly symmetrical portions, the intersections of these planes are principal axes. This follows readily from what has just been said of the principal axes of a rectangle; it appears also from the consideration that each plane will in that case contain two of the geometrical axes of the central ellipsoid.

If \( APB \) be an ellipse (fig. 23.), the principal axes of \( G \) are \( GA, GB, \) and a line perpendicular to the plane \( APB, \)

\[
B = \rho \int x^2 \sqrt{a^2 - x^2}
\]

\[
= 2 \rho \frac{b}{a} \int x^2 \sqrt{a^2 - x^2}
\]

\[
= 2 \rho \frac{b}{a} \left\{ -\frac{1}{3} (a^2 - x^2)^{\frac{3}{2}} - x \left( a^2 - x^2 \right)^{\frac{1}{2}} \right\},
\]

\[
\frac{1}{3} B = 2 \rho \frac{b}{a} \left\{ \frac{1}{3} \int x^2 \sqrt{a^2 - x^2} \right\};
\]

\[
\therefore \frac{4}{3} B = 2 \rho \frac{b}{a} \left\{ \int_{-a}^{a} x^2 \sqrt{a^2 - x^2} - \frac{1}{3} x \left( a^2 - x^2 \right)^{\frac{3}{2}} \right\};
\]

\[
\therefore B = \frac{1}{2} \rho \frac{b}{a} \left\{ a^2 \left( \text{circular area} \sin^{-1} \frac{x}{a} \right) - x \left( a^2 - x^2 \right)^{\frac{3}{2}} \right\} + C,
\]

which from \( x = -a \) to \( x = +a \)

\[
= \frac{1}{2} \rho \frac{b}{a} \left( a^2 \cdot \frac{\pi a^2}{2} \right) - c \frac{\pi a^3 b}{4}
\]

\[
= \mu \frac{a^2}{4},
\]

and \( A = \mu \frac{b^2}{4} \); \( \therefore \) \( C = \mu \frac{a^2 + b^2}{4} \).
The principal moments for a circle are immediately deducible by making \( b = a \), as are those for an elliptic or circular cylinder by multiplying by the height.

We may however obtain \( C \) more readily for a circle.

For if \( GR \) (fig. 24.) = \( r \),

\[
r^2 = x^2 + y^2 = d_{\mu} C, \quad \text{and} \quad d_{\mu} = 2\pi r \cdot \rho;
\]

\[
\therefore \quad C = 2\pi \rho \int r^3 = \rho \frac{\pi a^4}{2} = \mu \frac{a^2}{2}, \quad \text{if} \quad GA = a.
\]

Every radius \( GA \) is a principal axis;

\[
\therefore \quad A = B = \frac{C}{2} = \mu \frac{a}{4}.
\]

For a solid of revolution about the axis of \( x \),

\[
d_x \mu = \pi r^2 \cdot \rho = \pi (fx)^e \cdot \rho;
\]

\[
\therefore \quad C = \pi \rho \int (fx)^e.
\]

And \( A = B \);

\[
\therefore \quad d_{\mu} A = \frac{1}{2} (d_{\mu} A + d_{\mu} B) = \frac{1}{2} (x^2 + y^2) + x^2
\]

\[
= \frac{1}{2} r^2 + x^2;
\]

\[
\therefore \quad A = \frac{\pi \rho}{2} \int (fx)^e + \pi \rho \int (fx)^e \cdot x^2.
\]

If \( G \) be the centre of an ellipsoid (fig. 25.) the axes \( GA, GB, GC \) are principal axes;

and \( d_x A = \rho \int_y (y^2 + x^2) \).

But if \( PNP' \) be a section at the distance \( GM = x \), the equation to it is

\[
\frac{y^2}{b^2} + \frac{x^2}{a^2} + \frac{z^2}{c^2} = \frac{1}{1 - \frac{x^2}{a^2} \cdot \frac{1}{1 - \frac{a^2}{a^2}} = 1;}
\]
\[ MN = b \sqrt{1 - \frac{x^2}{a^2}}, \quad MP = c \sqrt{1 - \frac{x^2}{a^2}}, \]

and \( \rho \int y \int (y^2 + z^2) \) between the limits
\[ y = -MN, \quad z = -MP, \]
\[ y = +MN, \quad z = +MP, \]
is manifestly the moment of inertia of the plane ellipse \( PNP' \) about \( GA \);

\[ d_A \mu = \frac{\pi \rho}{4} bc \left( 1 - \frac{x^2}{a^2} \right) \left\{ b^2 \left( 1 - \frac{x^2}{a^2} \right) + c^2 \left( 1 - \frac{x^2}{a^2} \right) \right\}; \]

\[ A = \frac{\pi \rho}{4} bc (b^2 + c^2) \left\{ x - \frac{2}{3} \frac{x^3}{a^3} + \frac{1}{5} \frac{x^5}{a^5} \right\} + C, \]

which from \( x = -a \) to \( x = +a \)

\[ = \frac{4\pi \rho}{15} abc \cdot (b^2 + c^2) = \mu \cdot \frac{b^2 + c^2}{5}. \]

In all the above cases the moment of inertia is of the form \( \mu k^2 \), where \( k \) is a constant quantity. And this will be the case in any system whatever, since the moment is made up of positive products \( m (x^2 + y^2) \) each of which is of that form.

The line \( k \) is called the radius of gyration.

If \( G \) (fig. 26.) be the centre of gravity of a free rigid system, \( k \) the radius of gyration at \( G \), and \( Gg \) in the plane of the paper a principal axis perpendicular to \( GO \), an impulsive force \( S \) applied in a direction perpendicular to the plane of the paper at a point \( G \) whose distance from \( G \) is \( s \), is equivalent to a force \( S \) applied at \( G \) and a couple \( S's \) in a plane perpendicular to \( Gg \).
The effect of the former is to produce a motion of translation in the direction of \( S \). Let \( v \) be the velocity generated, then \( S = \mu v \).

The effect of the latter is to produce a rotatory motion round \( GC \). Let \( \omega \) be the velocity of rotation,

then \( S \cdot s = \omega \mu \kappa^2 \).

Let \( C \) be a point such that \( GC \cdot \omega = v \), then \( C \) and every point in a line through \( C \) parallel to \( Gg \) will remain at rest; therefore this line will be the axis about which the body or system will actually turn during the first instant in absolute space.

The line thus obtained by reversing the process in pp. 76, 77, and compounding the motion of translation with the rotatory motion is called the *Spontaneous Axis of Rotation* (v. p. 25.), which is therefore a principal axis at a point \( C \) which is sometimes called the *Centre of Spontaneous Rotation*, and whose position in \( OG \) produced is determined by the consideration that

\[
GC \cdot \frac{S \cdot s}{\mu \kappa^2} = \frac{S}{\mu} \quad ; \quad GC = \frac{\kappa^2}{s}.
\]

If the axis \( Cc \) were fixed, the shock of any force \( S \) on \( O \) would evidently produce no pressure whatever on \( Cc \). Hence \( O \) is called the *Centre of Percussion* corresponding to the axis \( Cc \). Its position is determined from the equation

\[
GO = \frac{\kappa^2}{CG}.
\]

If a rigid body or system turn about any fixed axis \( Cc \) with an angular velocity \( \omega \), the motion at any instant is equivalent to a simple rotatory motion about an axis \( Gg \) through the centre of gravity parallel to \( Cc \), and a couple of rotatory motions whose moment is \( CG \cdot \omega \); that is, a motion of translation with a velocity = \( CG \cdot \omega \); for these are the motions into which, if the axis were set free, it would immediately be resolved.
Now the motion of $G$ is unaffected by the former, and the latter would be generated by a force $S$ applied at $G = \mu \cdot CG \cdot \omega$.

Therefore the resultant couple of the effective forces in a plane perpendicular to $Cc$

$$= \omega \cdot (\text{moment of inertia round } Cc)$$

$$= \frac{S}{\mu \cdot CG} (\mu k^2 + \mu CG^2) \quad (\text{page 83.})$$

which, if $O$ be a point such that $GO = \frac{k^2}{CG}$, $= S \cdot CO$.

But a force $S$ applied at $O$ is equivalent to a force $S$ at $C$, which would be destroyed by the reaction of $Cc$, and a couple whose moment is $S \cdot CO$.

Hence a rotatory motion about a fixed axis $Cc$ with a velocity $\omega$ would be produced by a force $S = \omega \cdot \mu \cdot CG$ applied at $O$.

But since the axis is fixed, the same effect would be produced by a force $R$ applied at $G$

$$= S \cdot \frac{CO}{CG} = \omega \cdot \mu \cdot CO.$$ 

Now if the whole mass were collected at $O$ and connected with $C$ by an imponderable rigid rod, the force which must be applied at $O$, which would then become the centre of gravity, to cause $\mu$ to revolve about $C$ would be

$$\mu \times (\text{linear velocity}) = \mu \cdot \omega CO$$

It is evident that the velocity of the particle $m$ at $O$ is the same in either case.

Any point $O$ lying in a cylindrical surface at a distance $CG + \frac{k^2}{CG}$ from $Cc$ is called a centre of oscillation corresponding to the axis $Cc$. 

\}
The angular velocity due to a force \( R = \frac{R}{\mu \cdot CO} \), and the circumstance of the axis being fixed enables us to replace any force \( T \) acting at \( Q \) perpendicular to the plane of the paper by a force \( R = T \cdot \frac{CQ}{CG} \) at \( G \), parallel to \( T \).

The same reasoning holds for a succession of impulsive forces, and thus for a continued force, such as gravity.

If \( CA \) (fig. 27.) be a vertical, \( ACG = \theta \), the impulsive force at \( G \), which acts at every successive instant during the short time \( \Delta t \) to produce an angular velocity \( \Delta \omega \),

\[
= \mu g \sin \theta.
\]

And each successive increment of velocity being independent of the former, the whole increment \( \Delta \omega \) in the time \( \Delta t \)

\[
\Delta \omega = \frac{\mu g \sin \theta}{\mu \cdot CO};
\]

\[
\Delta \omega = \frac{g \sin \theta}{CO};
\]

\[
\therefore \text{ taking the limits } d\omega = \frac{g \sin \theta}{CO}.
\]

But \( \Delta \theta \), the angle described in any small time \( \Delta t \) during which \( \omega \) may be considered uniform,

\[
= \omega \cdot \Delta t;
\]

\[
\therefore d\omega = \omega;
\]

\[
\therefore d^2\theta = d\omega = \frac{g \sin \theta}{CO}.
\]

The same expression for determining \( t \) which would have arisen in considering the motion of any mass suspended from \( C \) by an immaterial thread \( CO \).
III. *Conservation of Couples. Equations of Euler.*

The following statement and demonstration of the principle of the Conservation of Couples are translated from the Memoir presented by M. Poinsot to the Institute in May 1804.

Let any number of perfectly free bodies, unconnected with each other, describe uniformly straight lines in space; then the forces by which they are respectively impelled remain always the same and in the same direction.

Therefore the resultant of all these individual forces passing through any fixed point, and the resultant couple, are the same at every instant during the whole motion.

Now if we suppose these bodies to be suddenly connected together so as to act one upon another by virtue of any reciprocal forces whatever, that is to say, such that between any two bodies the action and reaction are perfectly equal and opposite, which comprehends all forces of this nature, the individual motions of the several bodies will be changed, and the forces which respectively impel them will vary in magnitude and direction during every instant of the motion. But the resultant of these forces passing through any fixed point and the resultant couple will remain the same as before, and would still be the same if the bodies were suddenly set free, and each were to fly off in a straight line with the new velocity by which it is actually impelled.

This principle, which follows from the Differential Equations of Motion, may also be demonstrated in the following manner.
Since each body by reason of its connection with the others is unable any longer to obey fully the impulse which it has received, the force which acts on it is decomposed into two others, one of which is destroyed, while the other is that which the body actually obeys. The circumstances are identical with those of the case in which a body in motion meets with an insurmountable obstacle, except that the component part of the force which would then be annihilated, proceeds to act upon the other bodies of the system, and is destroyed by the united action of the similar components which each of them contributes.

Now it is evident that the resultant force passing through any fixed point, and the resultant couple, of the impulsive forces are respectively equivalent to the resultant forces and resultant couples of the two sets of forces into which they are decomposed. But the first set being in equilibrio their resultant force and resultant couple vanish by the laws of Statics: therefore the original resultant force and resultant couple are identical with those which the body actually obeys. With respect to any other forces, such as mutual attractions, which may exist in the system, since they are reciprocal, that is to say, distributed in pairs of equal and opposite forces, they cannot in any way affect the forces above mentioned.

We see therefore that in a system of bodies which have received any primitive impulses and which act mutually upon each other in any manner, the resultant of all the forces which impel them passing through any fixed point, and the resultant couple, remain always the same whatever variations the respective moving forces of individual bodies may experience, whether these variations take place by insensible degrees, or abruptly, from any change in the reciprocal actions of the bodies, or from the sudden introduction of any new connecting forces among them.
Hence in note (16) the resultant of all the effective couples at any instant during the motion = $M$.

The centrifugal couple is of course included. It is the portion of the resultant $M \sin \theta'$ of the couples whose planes pass through $GP$, which lies in a plane passing also through $GM$.

The projections of $GM = M$ (fig. 12.) on three fixed lines $Gx, Gy, Gz$ in space, will represent in magnitude and direction the axes of the resultant couples of the effective forces in planes perpendicular to these lines.

Let $a, b, c$, be the cosines of the angles which the axes of the ellipsoid make with the line $Gx$, then

$$M \cos MGx = M \cos a \cdot a + M \cos \beta \cdot b + M \cos \gamma \cdot c$$

$$= A \omega_a \cdot a + B \omega_b \cdot b + C \omega_c \cdot c.$$

Similarly

$$M \cos MGy = A \omega_a \cdot a' + B \omega_b \cdot b' + C \omega_c \cdot c',$$

$$M \cos MGz = A \omega_a \cdot a'' + B \omega_b \cdot b'' + C \omega_c \cdot c''.$$


It must be observed that the portion of the centrifugal couple, as obtained in note (15), which lies in the plane perpendicular to the axis of instantaneous rotation is omitted in deducing the final equation, as having no tendency to change the position of the axis.

IV. *Application to the Precession of the Equinoxes.*

Let $G$ (fig. 28.) be the Earth's centre, $GC$ its geometrical axis; $S$ the Sun; $S'LW$ the equator, the Earth being supposed in the position which it has at the
summer solstice. Then the action of the Sun on $S'$ is greater, and on $W$ less than the action on $G$. Therefore in addition to the force on $G$ which produces the motion of translation there is a couple of forces in opposite directions which produces a rotatory motion round the line $\varphi L$ perpendicular to the plane $KGS$. Also in winter the opposite face being presented to $S$ the couple tends to produce motion in the same direction. If therefore the Earth when originally projected had a rotatory motion of its own about $GC$, which would be the case if the primitive impulse did not pass through $G$ but through some other point in the equatorial plane in a direction perpendicular to a plane passing through $GC$, these two rotatory motions would be compounded, and the pole of rotation on the central ellipsoid would be drawn aside to a short distance from the geometrical pole. It would therefore describe as its poloid a circle at this distance from the pole $C$. The serpoloid would also be a circle about $GK$. For the plane of the resultant couple would be parallel to the plane of the ecliptic.

The ellipticity of the Earth being very small the effect of the centrifugal forces is not perceptible.

When the Earth is in any other position the effect of the Sun's attraction is at some times to increase and at others to diminish the obliquity, which however it does not permanently alter.

To determine the velocity of the Pole we must know the amount of the effect of the disturbing couple at $S'$ and $W$, which depends on the Sun's attraction at those points. (v. Airy, Precession, Art. 21.)

If $x$ be the angular velocity generated by the couple at $S'$ and $W$, $\omega$ the angular velocity of the Earth's rota-
tion round the instantaneous axis which is always the same, \( \rho \) the radius of the poloid,

\[
2\pi\rho = (\text{velocity of the Pole}) \times (\text{one day}).
\]

But by note (26.), vel. of the Pole = (radius of \( \oplus \)) \( \frac{\alpha}{\omega} \),

and length of a day = \( \frac{2\pi}{\omega} \);

\[
\therefore \quad \rho = (\text{radius of } \oplus) \cdot \frac{\alpha}{\omega^2}. \quad (v. \text{ Airy, Art. 13.})
\]

The value of \( \rho \) is obtained by observation in note (9).