CHAPTER IX.

PROBLEMS IN MACHINE DYNAMICS.

§ 44.—TRAIN RESISTANCE.

IN the commencement of the last chapter it was pointed out that as long as a body was not actually changing its form it was said to be *in equilibrium*.

Further, we have seen that although any force, however small, inevitably alters the form of the bodies, however rigid, upon which it acts, yet that these alterations of form are, in a machine, intentionally made so minute that we are able to neglect them, and to treat the various bodies forming part of the machine as if their forms really remained unchanged.[1] We called the conditions of equilibrium **static** in the case where there were acting only external forces and pressures[2] in addition to the stresses in the links. This, we saw, corresponded to a condition either of rest or uniform velocity on the part of each body in the machine. We have examined in the last chapter most of the principal problems connected with the static equilibrium of bodies having constrained motion. In the present chapter we shall examine

[1] We neglect here the case of springs, leather belts, and the one or two other instances in which the alteration of form under external force is comparatively great.

[2] See § 36, p. 263.

a number of problems of a kind often distinguished from the former as dynamic instead of static, but which it is perhaps better to distinguish as problems of **kinetic**, instead of **static, equilibrium**. The difference between the two conditions is not the difference between no force and force, or between rest and motion, but between rest *or* motion with uniform velocity, and motion with varying velocity, that is, with acceleration. In this case not only external forces and pressures act upon the links, but also resistances due to their acceleration, which resistances may be positive or negative according to the sense of the acceleration. These resistances are forces whose magnitudes depend upon the *masses* of the links themselves, and are in this respect essentially different from the forces and pressures previously considered, in working with which the masses of the bodies acted upon never entered into the question. It is on this account that we separate the problems in which they occur from those formerly considered, and not because both sets of problems do not alike relate to conditions of equilibrium.

In the present chapter we shall examine in detail certain typical and very important problems of kinetic equilibrium representing those which have actually to be solved in engineering work. We shall take first in this section, as perhaps the simplest case that can be taken, the case of the motion of a train, from which we have already used several illustrations in § 28. We shall next examine dynamically a simple direct-acting pumping engine, then a Cornish engine, then the driving mechanism of an ordinary horizontal steam engine, next the fly-wheel of such an engine, and its connecting rod, and lastly the case of centrifugal governors.

Let us suppose that a train weighing 80 tons, or say 180,000 pounds, is running on a level at 40 miles per hour, and that 40 wheels are simultaneously braked, with such a pressure on each brake as gives a frictional resistance of 30 pounds at the periphery of each wheel. The total brake pressure is thus $40 \times 30 = 1200$ pounds. The speed of the train is 58·4 feet per second. It is required to draw a diagram of the stop of the train.

It may be stated at once that in such a very simple case as this one, diagrams are by no means necessary, nor is a graphic solution to be preferred to ordinary calculation if only final results are required. Both calculation and diagram will be given here, the latter partly for the sake of the determination of its scales, and partly because by its nature it gives not only a final result—the distance run before stopping, or whatever it may be—but also a pictorial representation of the whole process of stopping, which it is in many cases important to follow, and which otherwise can only be understood by a series of separate calculations, together much more trouble than the drawing of the diagram.

The distance which will be run before stopping can be found at once by calculating the kinetic energy stored up in the train, as it moves with the given velocity, and dividing this quantity by the resistance to its forward motion, namely, the resistance of the train proper plus the added artificial resistance of the brakes. The stored-up energy is

$$\frac{180{,}000 \times 58{\cdot}4^2}{64{\cdot}4} = 9{,}530{,}000 \text{ foot-pounds.}$$

The normal resistance of the train on a level may be taken as 8 pounds per ton, or 640 pounds. The brake resistance is 1,200 pounds, but it has to be overcome through a distance π times as great as that moved through by the train as a whole. The total resistance is therefore $640 + (\pi \times 1200)$ = (say) 4,400 pounds. The distance which the train will run before stopping is therefore $\frac{9{,}530{,}000}{4{,}400} = 2{,}170$ feet, or about two-fifths of a mile.

The diagram (Fig. 148) which represents this case is drawn with the following scales :—

(1) Force or pressure scale 2,000 pounds = 1 inch.
(2) Distance scale 1,200 feet = 1 inch.
(3) Velocity scale 20·6 feet per second = 1 inch.
(4) Acceleration scale 0·356 foot-seconds per second = 1 inch.

The first two scales are taken arbitrarily as may be convenient. The acceleration scale is derived from the force scale by the relation $a = f\frac{g}{w}$ (p. 222), which tells us that 1 inch must stand for $\frac{g}{w}$ times as many units of acceleration as of pressure. This fraction is here ·000178, so that the acceleration scale is 2000 times this, as given above. The velocity scale is derived from the distance and acceleration scales also as before, one inch standing for $\sqrt{0{\cdot}356 \times 1200}$, or 20·6 feet per second (very nearly).

The distance OA is first set up for the initial velocity of the train, and ON for the acceleration (here negative, of course). AB_1 is drawn at right angles to NA. BB_1 is a vertical at any convenient distance from the origin. BN_1 is the acceleration (= ON), and the next segment of the velocity curve, B_1C_1, is drawn at right angles to N_1B_1 and so on. This gives the curve $AB_1C_1 \ldots S_1$, and the distance OS_1 run before the train stops (see p. 204). But the process is obviously one which gives cumulative errors, and these are too great to be neglected. For not only is the point C_1, for instance, too high in position on account of the substi-

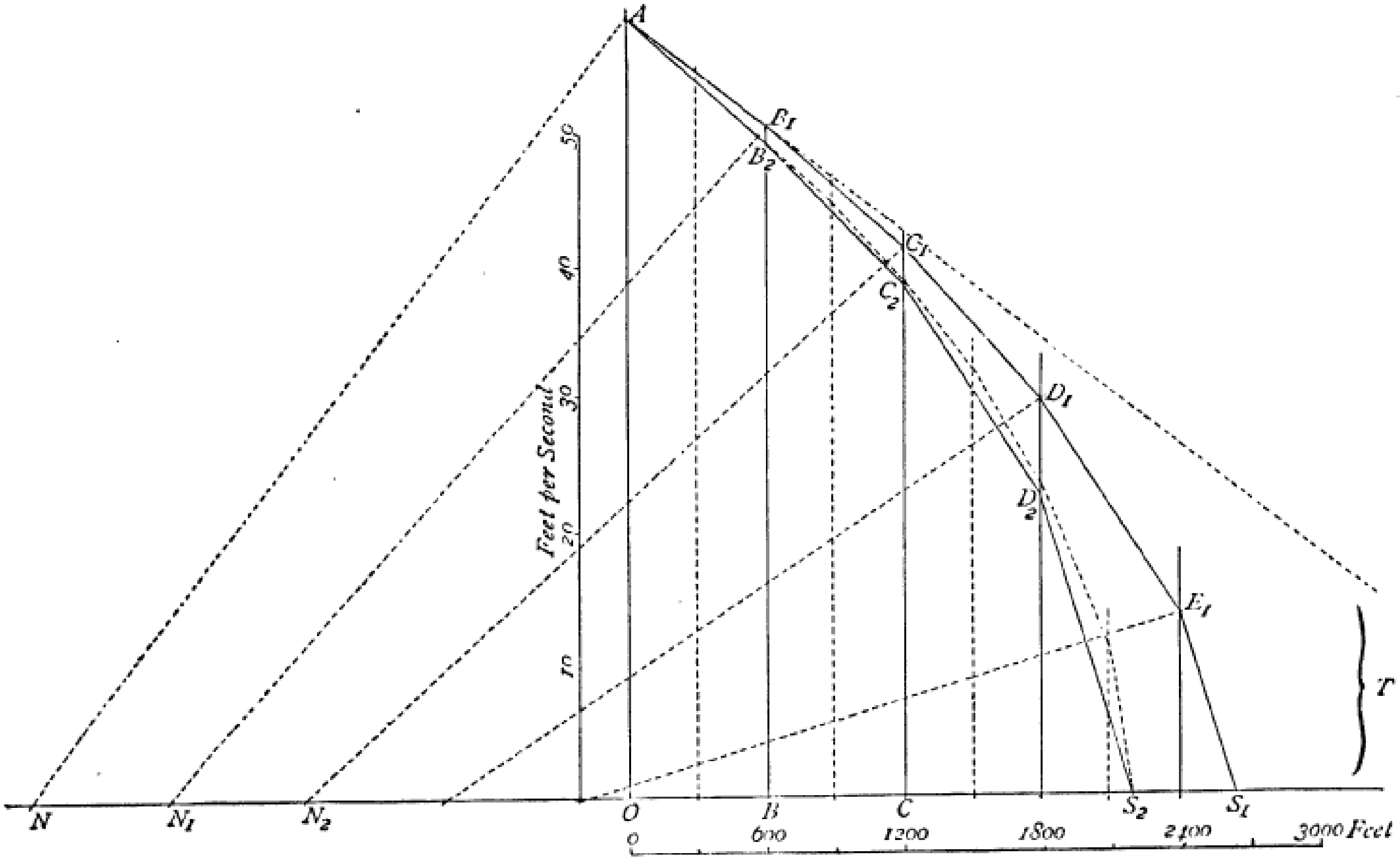

A
B1
B2
C1
C2
D1
D2
E1
Feet per Second
10
20
30
40
50
T
N
N1
N2
O
B
C
S2
S1
0
600
1200
1800
2400
3000 Feet

FIG. 148.

tution of the straight line B_1C_1 for an arc convex upwards, but it is further misplaced by the similar error in the position of B_1, which makes the angle BB_1C_1 too great. The error of each point in the curve is thus greater than that of the preceding one, and the whole distance OS_1 is (in this example) nearly 20 per cent. too great. The longer the distance intervals OB, BC, &c. be taken the greater does this error become. For any reasonable distance intervals, however, the point S_1 can be obtained very approximately by drawing a second polygon $AB_2C_2 \ldots S_2$, so that AB_2 is parallel to B_1C_1, B_2C_2 parallel to C_1D_1, C_2D_2 to D_1E_1 and so on. This construction makes the distance OS_2 in the figure very slightly too small. But although this error of S_2 is here quite negligible, yet with the distance intervals shown the intermediate points B_2C_2 and D_2 are very sensibly out of position. By taking the distance intervals sufficiently small, these points also can be brought sensibly right. In Fig. 148 the dotted curve is drawn like $AB_2C_2S_2$, but for distance intervals $= \frac{OB}{2}$ or 300 feet, and its points sensibly coincide with calculated points.

As the acceleration, which is the sub-normal to the velocity curve (p. 205), is constant, we know that the curve itself is a parabola, and as such it can easily be drawn. The point S_2 must lie midway between O and T, where AB_1 cuts the axis. In starting such a diagram the two quantities to be set out are the velocity and the acceleration. It is not really necessary, however, to set out the latter, for we know that it is proportional to the force, hence ON is really set out equal to the force (here resistance) causing (negative) acceleration. Knowledge of the scale on which ON represents the acceleration is only required in order to find on what scale the velocity must be drawn in order to correspond to our assumed distance scale, or—if the velocity scale has been arbitrarily assumed—to determine on what scale OD must be read of as distance.

Fig. 149 shows the same problem worked out on a time, instead of a distance, base. The velocity curve is here, as

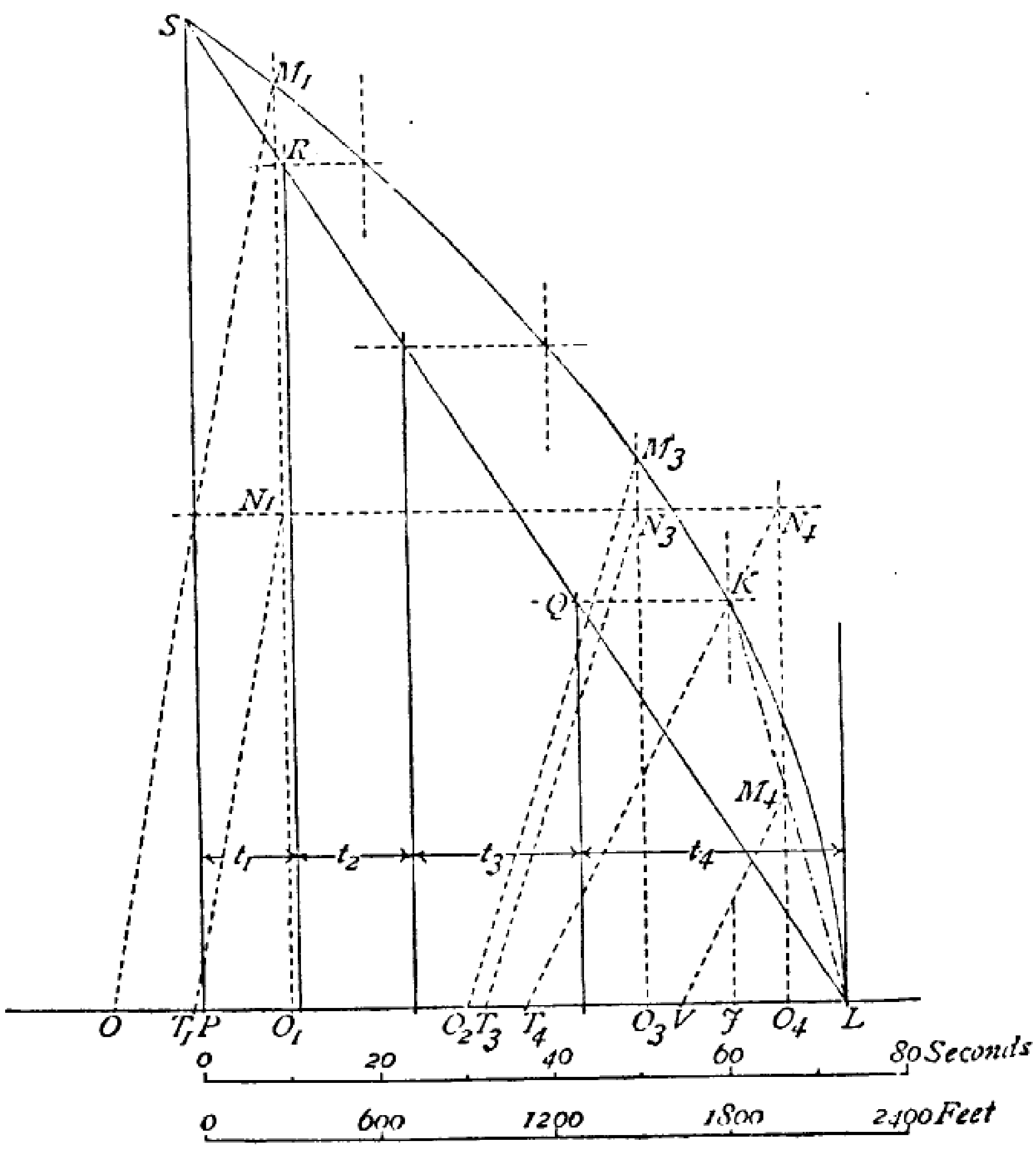


FIG. 149.

in Fig. 92, a straight line, the acceleration being constant. The acceleration is equal to $\frac{f}{m}$ or $f\frac{g}{w}$, which is here

$$4400 \times \frac{32 \cdot 2}{180{,}000} = 0 \cdot 787$$ foot-seconds per second. The

time of the stop (we have already measured its *distance*) is $\frac{58\cdot4}{0\cdot787} = 74\cdot2$ seconds, or about a minute and a quarter. The time scale in the figure is taken so as to make the total length of the base the same as in the former case (see § 28 p. 211). This necessarily makes the actual time scale an odd one. As the diagram is drawn $\frac{2,170}{1,200}$ inches stand for 74·2 seconds, so that the time scale is 41 seconds per inch.

A part of the construction, which is the same as that of Fig. 100, described on p. 209, is given in Fig. 149. SKL is simply a copy of the velocity curve of Fig. 148. O_1O is a length equal to one distance interval, set back from O_1, the centre of the first interval. O_1N_1 is a distance determined by the method of p. 209. In this case there are $\frac{20\cdot6}{1200} = 0\cdot0172$ as many units of velocity as of distance in one inch, so that $n = 0\cdot0172$. The distance O_1N_1 must therefore be made equal to $\frac{1}{0\cdot0172} = 58\cdot1$ time units. The time unit being (as found above) $\frac{1}{41} = 0\cdot0244$ inch, the length O_1N_1 must be $0\cdot0244 \times 58\cdot1 = 1\cdot42$ inches. N_1T_1 being drawn parallel to M_1O, the distance $O_1T_1 = t_1$ represents the time taken by the train in traversing the first distance interval, and R is therefore a point on the new velocity curve. Similarly $O_3T_3 = t_3$ represents the time taken in traversing the third distance interval, and Q is a point on the new velocity curve. This curve as a whole is simply the straight line $SRQL$, as in Fig. 92. It must not be forgotten that in getting the time $O_4T_4 = t_4$ for the last distance interval, the distance O_4V must be made equal to

JL, the actual length of that interval and set back from its mid point. It is better to take M_4 on the chord than on the arc *KL*.

In all these cases, as has been already pointed out, the diagram has been scarcely more than illustrative of the problem—the actual answer has in each case been worked out quite independently of it. With altered data, however, such as are quite likely to occur in practice, it is often convenient to use the graphic method for its own sake, although even then its comparative convenience is not nearly so great as in the other cases which we have to consider in this chapter.

Let us now suppose that the stop is not to be made on a level line, but on one which has first an uphill gradient of 1 : 250 for 1,000 feet, is then level for 500 feet, then has a down-hill gradient of 1 : 150. On the first section the resistance will be $\frac{180,000}{250} = 720$ pounds *more* than previously, or 5,120 pounds in all; on the second section everything will be precisely as before; on the last section the resistance will be $\frac{180,000}{150} = 1,200$ pounds *less* than before, or 3,200 pounds in all. These quantities would be used in the diagram as the sub-normals in the first, second, and third sections respectively. At the top of the hill the velocity of the train has been reduced to 39·7 feet per second. At the end of the level ground its velocity is 28·0 feet per second, while the energy still stored up in the moving train is 2,210,000 foot-pounds. With the now diminished negative acceleration due to the downward gradient, the train will still not come to rest for 690 feet, so that the whole stop will occupy 2,190 feet. As the (negative) acceleration is constant on each of the three sections, the velocity curve consists of arcs of three parabolas, all having the same axis. As the distance run in each case before stopping varies directly as the resistance and as *the square* of the initial velocity, it is especially important that the train should enter any section where the resistance is diminished (such as the down-hill section in the case supposed) with as small a velocity as possible.

We may take one more example in this section. Let it be supposed that a train of the same weight, &c., as before, is two miles from a station and running up a continuous incline of 1 in 250, in the middle of which the station is placed. How long will it take to reach the station, and how far would it over-run the station if the brakes "leaked off" suddenly after being on for 20 seconds? To answer the first question we have first to find when the brakes have to be applied in order that the train may stop just at the station. The total resistance of the braked train on an upward gradient of 1 in 250 is 5,120 pounds. The stoppage from a speed of 40 miles per hour under this resistance will require $\frac{9,530,000}{5,120} =$ 1,860 feet. Before applying the brakes the train will therefore have run 8,700 feet at 40 miles an hour, which will occupy 149 seconds. The negative acceleration, when the brakes are applied, will be $5,120 \times \frac{32 \cdot 2}{180,000} = 0 \cdot 916$ foot-seconds per second, and the duration of the stop itself therefore $\frac{58 \cdot 4}{0 \cdot 916} = 64$ seconds. The whole time taken up in running the two miles will be 213 seconds.

If the brakes leaked off suddenly at the end of 20 seconds the train would be left running, under its own proper resistance only, at a speed reduced by $0 \cdot 916 \times 20 = 18 \cdot 3$ feet per second. The speed of the train is therefore (say) 40 feet per second, the kinetic energy at this speed is 4,480,000 foot-pounds, and the resistance of the train, without the brakes, 640 + 720 = 1,360 pounds. The train will therefore run on $\frac{4,480,000}{1,360} = 3,290$ feet before it stops. When the brakes were applied the train was 1,860 feet from the station. In 20 seconds this distance must have been diminished by 984 feet, leaving the train 876 feet from the station, which it would therefore over-run by 2,414 feet, or nearly half a mile.

From these data there can at once be calculated the velocity with which the train would strike any obstacle which happened to be standing in its way at the station.

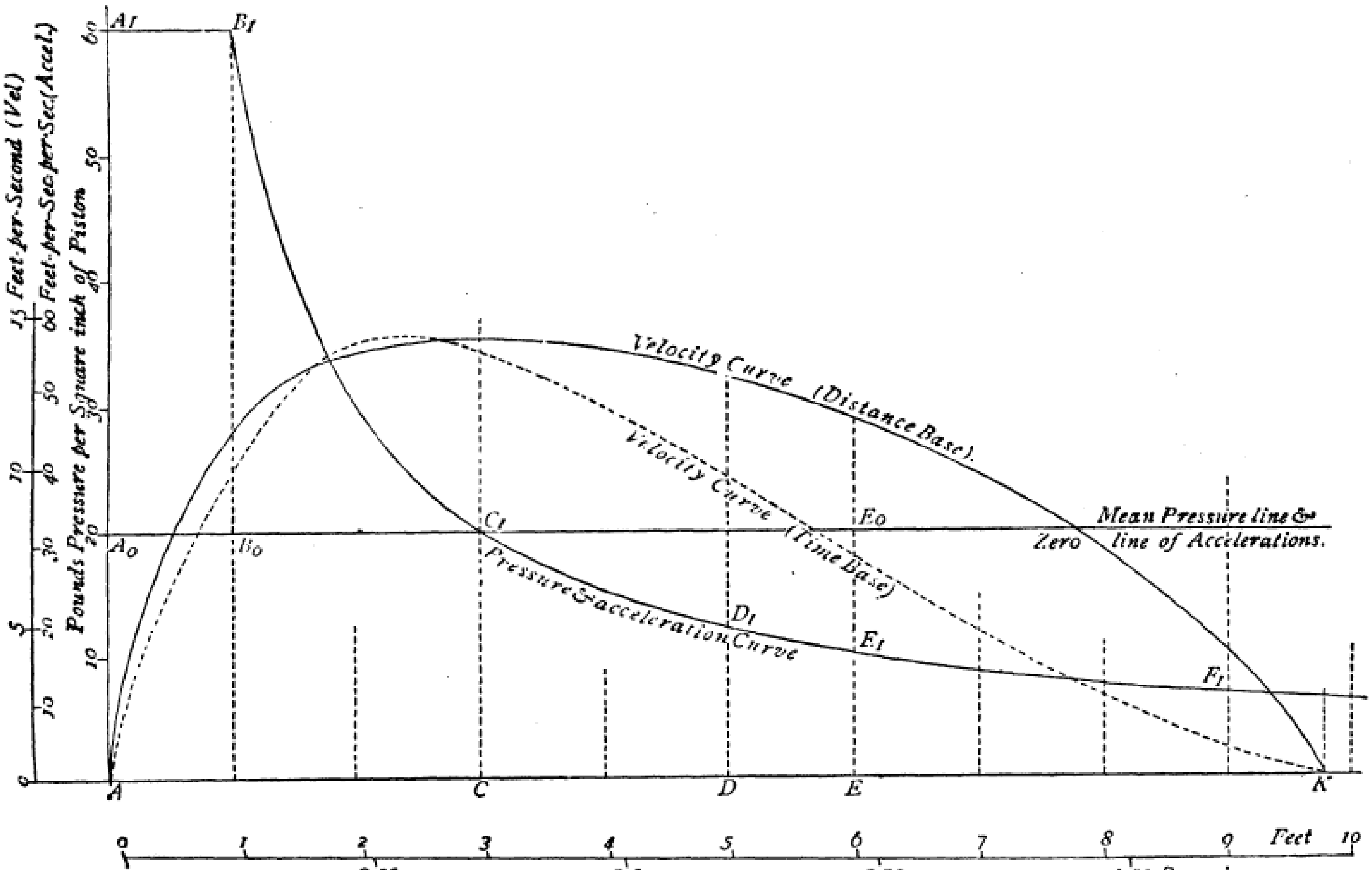
§ 45.—DIRECT ACTING PUMPING ENGINE.

In one of the simplest forms of pumping engines, known as the "Bull Engine," the cylinder is placed vertically above the pump shaft, and the pump rods or "pitwork" simply hung direct to the piston rod. Kinematically the combination is nothing but a sliding pair of elements, there being no crank or rotating parts of any kind. Dynamically the machine is of much more interest, and its action much more complex. For although the form of the piston and cylinder prevent any relative motions but those of an ordinary sliding pair, yet they do not in any way affect or control the velocity of motion, and the length of stroke of the engine is entirely dependent on the accelerating forces in action, and it both may and does vary from stroke to stroke instead of being an absolutely fixed distance as in an ordinary engine. Buffer blocks are provided for the crosshead to strike against at each end, but if the accelerating forces exceed certain limits these may be destroyed and the cylinder cover or end knocked out.

We shall investigate the condition of the working of an engine of this type by the aid of the principles and constructions already examined.

Let there be given, as in Fig. 150, a diagram $AA_1B_1C_1$ etc., whose ordinates represent the steam-pressures below the piston tending to lift it. The point at which this curve ends on the right we do not know, because the stroke of the engine is not a fixed quantity, but one which we have to find out. Further let there be given the weight of the whole pitwork, AA_0, which forms the resistance against which the piston has to rise, and which is constant throughout the stroke, whatever its length may be. The horizontal line $A_0 \ldots F_0$ will then form a resistance diagram. From these data our first problem will be to find the velocity at different points of the stroke, *i.e.*, to construct a curve whose ordinates shall measure these velocities.

The actual effort available for producing acceleration is at each instant the difference between the total effort and the resistance, as A_0A_1, B_0B_1, E_0E_1 etc. During the first part of the stroke these efforts are positive, and the speed of the piston will therefore increase. During the second part they are negative—the resistance being greater than the effort—and the acceleration is therefore also negative, the speed of the piston diminishing until at length it becomes zero, and the piston stops. The total resistance AA_0 is partly due to the gravity of the mass and is partly frictional, but in this case we may neglect this latter part, and treat the ordinate AA_0 as representing simply the weight of the pitwork. If we take the ordinate AA_1, (as is commonly done) to represent not the *whole* pressure on the piston, but the pressure *per square inch*, then AA_0 must represent also the resistance *per square inch*, *i.e.* the total resistance divided by the area of the piston. There is therefore a nett effort A_0A_1



15 Feet-per-Second (Vel)
60 Feet-per-Sec.per-Sec.(Accel.)
Pounds Pressure per Square inch of Piston
A1
B1
Ao
Bo
C1
D1
E1
Eo
F1
Velocity Curve (Distance Base).
Velocity Curve (Time Base)
Pressure & acceleration Curve
Mean Pressure line & line of Accelerations.
Zero
A
C
D
E
K
0
1
2
3
4
5
6
7
8
9
Feet
10
0·25
0·5
0·75
1·00 Second

FIG. 150.

per square inch of piston to be expended in accelerating a mass equal to AA_0, also per square inch of piston. This mass is constant throughout, so that the acceleration of the pitwork at each instant is simply proportional to the force producing it, that is, to the ordinate between the line $A_0 \ldots F_0$ and the curve $A_1B_1 \ldots F_1$, and is positive or negative according as that curve lies above or below the line $A_0 \ldots F_0$. The curve $A_1B_1 \ldots F_1$, may therefore be taken simply as an acceleration curve, *the scale of which is as yet unknown*, on a distance base, and the problem is the one treated already in Fig. 99, given an acceleration diagram on a distance base, to construct from it a curve of velocities. The construction, being exactly the same as that of Fig. 99, is not repeated here,—the velocity curve is shown in the figure. The velocity reaches a maximum where the acceleration is zero (at C_1) and then diminishes under the negative acceleration until at K the curve cuts its axis. At this point therefore, motion ceases, and the stroke is completed, its length being AK, or 9·8 feet on the scale used.

If there has been room for this stroke in the cylinder, the piston will simply have come to rest clear of the cylinder cover by a certain distance. But our construction has obviously been quite independent of any particular length of cylinder. The piston would naturally come to rest at K, no matter how long the cylinder was, and if the space in the cylinder available for its motion were *less* than AK,—say AD,—the piston would strike the cylinder end when it had travelled so far, and the cylinder end would be broken unless it were strong enough to stand the blow. As such an accident would be very serious in an engine, it is guarded against, as mentioned above, by placing buffers or buffer beams of some description outside the cylinder in such a position that the crosshead must strike them before the