CHAPTER VIII.

STATIC EQUILIBRIUM.

§ 35.—CLASSIFICATION OF THE FORCES ACTING ON A MECHANISM.

ON every link in a mechanism, including, of course, the fixed link, there are usually a number of forces acting. We shall find it possible to classify the possible force conditions as simply as we have classified the possible conditions as to velocity and acceleration.

In the first place, the whole forces acting upon and between the links of any mechanism can be divided into two classes, between which it is quite easy to distinguish. Some of the forces, namely, are entirely external to the mechanism itself, and both in direction and magnitude may be independent of it; such forces are called **external forces.** The weight hanging from a crane, the resistance of a piece of iron to the edge of a cutting tool, the pressure of steam on a piston, are examples of such external forces. The weights of the individual links in any mechanism also fall into this category.

When any such external forces act at different points upon a mechanism—whether or not they cause the mechanism to move—they give rise to other forces acting from link to link of the mechanism, determined in magnitude by the external forces, but fixed in direction solely by the nature of the mechanism itself. These forces, in default of a better name, we shall call **pressures.** It would be misleading to call them "internal" as opposed to the "external" forces, for although they are internal in respect to the mechanism as a whole they are external to its links individually. In such an example as that of Fig. 103 the *external forces* are the two weights W_b and W_d, whose magnitudes and directions may be anything whatever[1] that we choose. Besides these there are in the mechanism forces

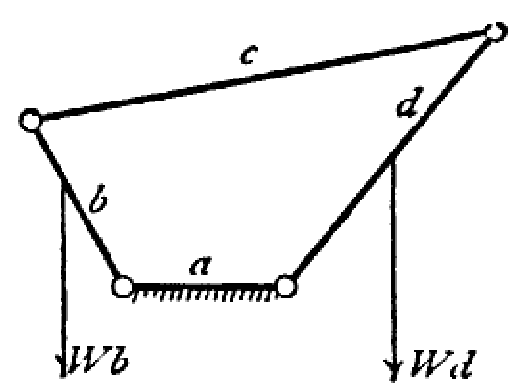

FIG. 103.

exerted by each link upon those next adjacent to it, whose total magnitudes are determined by the external forces, but whose directions and relative magnitudes are fixed by the mechanism itself. These are the *pressures*. The pressures exerted by *b* and *d* upon *c* are forces external to *c*, although not external to the mechanism. Similarly the pressures exerted by *a* and *c* upon *b* are external to it just as much as the external force W_b. But again they are not external to the mechanism, and therefore do not receive the name of external forces.

Pressures being by definition actions between adjacent links, occur always at the surfaces or lines of contact of the pairs of elements. We might say that they occurred always

[1] "Anything whatever," because we make no pre-supposition that the mechanism shall be "balanced" under them.

at the *joints* if we had only to do with turning pairs, but sliding surfaces are not generally included under the head of joints, although they are equally important to us as pairs. But it is clear that the pressures acting at different points, perhaps points very far apart, on the links, must be transmitted from point to point through the material of the links themselves. These transmitting, molecular forces might correctly be called internal forces; they have, however, received the more convenient name of **stresses**, which we shall always use to designate them.

In rigid bodies, stress may be defined as **resistance to alteration of form**; in fluids—which occasionally form part of machines, (see p. 3)—as **resistance to alteration of volume.** It is this capacity of the material of the links to exert stress in such fashion as to preserve their forms sensibly unaltered that justifies us in treating the virtual centre as a fixed point (p. 44). Could the shape of the links alter sensibly, the position of the virtual centre would be to a corresponding extent variable, the machine would become useless for its own proper purposes, and our method of examination would become inapplicable.

So long as the links of a mechanism are either stationary or moving with constant velocity, there come into question only these three—external forces, pressures, and stresses. Pressures act on each link from the next one at every pairing. The pressure of the link *a* on the link *b* is external to *b*, the pressure of *b* upon *a* at the same place is external to *a*, and so on. But no pressures can exist unless in the first instance they are called into existence by external forces acting on the mechanism. For the pressures may be taken to represent the resistance of the links (consequent on the manner in which they are connected together) to change of relative position, just as the stresses represent the resistance of the molecules (consequent on the manner in which *they* are connected) to change of relative position. The difference between the two cases is that the links are allowed to change their positions to a very large extent, and the molecules only to a very small one. The external forces may act on only one link of the whole mechanism, or on all, or on any number of the links. The mere weight of the links themselves may form a most important part of these forces, or may (as in a horizontal steam engine) be fairly negligible in comparison with the rest. The stresses, as representing entirely intermolecular action, may be left out of account here, it being presupposed only that the links are made of such material and dimensions as will keep the stresses in them so small that their change of form under pressure may be safely neglected. The stresses will then stand in the same general relation to the pressures that the pressures do to the external forces, except that wherever an external force acts on a link along with pressures it takes exactly the position of a pressure in causing stress.

The last paragraph has contained a general statement of the relations between stress, pressure, and external force in the case of bodies stationary or moving with constant velocity. When a body has acceleration, a force not falling properly under either of these three heads has to be taken into account. A body offers no resistance to continuance of motion in its own direction with its own velocity, but it cannot be accelerated without the action of force. This fact, which is Newton's "first law," is at the foundation of our whole study of dynamics. But it involves directly the converse fact that every body simply in virtue of its existence offers resistance to acceleration. This resistance is exactly measured by the force necessary to cause the acceleration, is equal and opposite to it, stands to it, in fact, in the relation of reaction to action. Neither can exist without the other; either may be looked at alone, but only if we do not forget that it is only half of a duality.[1]

When, therefore, any link of a mechanism is undergoing acceleration, its **resistance to acceleration**—a quantity proportional directly to its mass, as well as to the acceleration, but for which, unfortunately, we have no single word—is a force which has to be taken into account along with the rest, and which falls neither into the class of external forces nor into that of pressures, as we have defined them. We shall find presently that problems involving "resistance due to acceleration" are not more difficult to deal with than any others.

§ 36.—EQUILIBRIUM—STATIC AND KINETIC.

So long as the form of a body is not actually undergoing change—lengthening, shortening, distorting, etc.—the body is said to be in **equilibrium.** This equilibrium is called **static** if the body is stationary or moving with uniform velocity, and **kinetic** if it is undergoing acceleration.

For a body to be in static equilibrium it is necessary simply that the external forces acting upon it should not be such as could, in their united action, cause acceleration. Now the united or total action of any system of forces on a body is in every respect, except as to the stresses caused by the forces, the same as the action of the resultant or sum of that system of forces. The sum of any number of forces

[1] We talk similarly of the pressure of a girder on its abutment or of the reaction of the abutment against the girder. Neither can exist without the other, but without losing sight of the duality we often for simplicity's sake speak of only one.

may be either (i) zero, (ii) a single finite force of definite direction and position, or (iii) a couple, which has sense and has also magnitude measured as a moment, but has neither magnitude as a force nor any position or direction. So long as the forces are all in one plane, the condition always presupposed in this part of our work, no other condition than one of these three is possible.

If the sum of all the external forces be zero the body must be in static equilibrium, for zero force must cause zero acceleration.

If the sum of all the external forces acting on any link of a mechanism be a single force the equilibrium of the link is static or kinetic according to the position of that force. If the force passes through the virtual centre it can give the body no acceleration, because that point is a fixed one; no force whatever by acting on it can either make the body move or change its motion if it is already moving. In every other case a single force can and must cause the body to be accelerated. This may be summed up by saying that **if the sum of the external forces acting on any link of a mechanism be a single force, the link will be in static equilibrium only if that force act through its virtual centre relatively to the fixed link.**

If the sum of all the forces acting on any link of a mechanism be a couple, the condition of the link depends on the position of its virtual centre. If the link has a motion of translation only it will be in equilibrium, because its virtual centre is at infinity; in all other cases it must be undergoing acceleration. For looking at a couple merely as two equal, parallel, and opposite forces, there is no difficulty in seeing that it cannot cause acceleration in a body whose only possible motion is one of translation in one particular direction. For the two forces which together constitute the couple being parallel and opposed to each other, as well as equal, have no tendency to shift the body as a whole in the only way in which (by virtue of its connection with the rest of the mechanism) it can be shifted. Whatever motion either one of them could give it is directly counteracted by the opposing action of the other. A simpler proof of this is derived from the modern treatment of a couple as an infinitely small force at an infinitely great distance, acting along "the line at infinity." Such a force must (by definition) pass through all points at infinity, and therefore through that particular point which is the virtual centre for the motion of the link. The case therefore falls within that of the last paragraph—such a force can give the body no acceleration.

If, on the other hand, the virtual centre of the link be at a finite distance only, the link cannot be in static equilibrium, for the couple can cause angular motion, and therefore acceleration, although it cannot cause simple translation. This may be proved in various ways. In the first place, the last method of the last paragraph shows us that the couple is a force *not* now acting through the virtual centre, and therefore capable of causing acceleration. Or, secondly (without making use of the infinite elements), we know that a couple may be shifted anywhere in its plane of action without altering it, *i.e.*, that it has no special position. Therefore we may suppose it so shifted that one of the two forces of which it consists passes through the virtual centre. This force can therefore give the body no acceleration, but it leaves it under the sole action of the second force, which by hypothesis can *not* pass through the virtual centre. So far as acceleration goes, the body is therefore just in the position of one acted upon by forces whose sum is a single force. The two last paragraphs may be summed up as follows:—**If the sum of the external forces acting on any link of a mechanism be a couple, the link will be in static equilibrium only if its constrained motion be one of translation.**

When a body is in static equilibrium the external forces acting on it are generally said to be **balanced**. This expression may mean either of two different things. If the static equilibrium results from the sum of the forces being zero, it means that they are balanced among each other—that as a whole they have no tendency to move the body, because as a whole they have no magnitude. This condition occurs in structures constantly, but only extremely seldom in machines. If the equilibrium results from the sum of the forces passing through the virtual centre, the forces are *not* balanced among themselves—(for their sum is a finite force)—but are balanced by the pressures between the links of the mechanism which keep the virtual centre stationary.

When, however, the body is in kinetic equilibrium, the force (or couple) causing acceleration is often said to be **unbalanced**, and indeed the condition is not always recognised as one of equilibrium at all. But the force here, which is the sum of all the external forces in action, is balanced just as completely as the force passing through the virtual centre in the case just dealt with. Neither is balanced by what may be called **independent** as distinguished from **derived** external forces, and therefore in one sense both might be called unbalanced. But one we have just seen to be balanced by the pressures acting between the links, called into action inevitably, each in its particular direction, as soon as the external forces begin to act (p. 261), and the other (with which we are now dealing) is no less really balanced by the resistance to acceleration (p. 263). What is generally called an unbalanced force, or couple, is therefore one which is not balanced either by external force or by pressure, but solely by the resistance of the body or bodies upon which it acts to acceleration. There is no particular harm in the use of the word unbalanced in this case, if this limitation of its meaning be kept in mind.

In speaking first of force (p. 217) we said that we measured and compared the magnitudes of forces only by the accelerations they produced or could produce. We find now that a body may be acted on by forces to any extent and yet be undergoing no acceleration. There is here, however, nothing contradictory. The conclusion which we have to draw when we see forces acting on a body which stands still or moves with constant velocity, is that the accelerations produced by those forces must be such as exactly to counteract each other, so that as regards acceleration the state of the body is the same as if no force were acting at all. It is because we know that in such a case the body receives no acceleration that we say that the forces acting on it must be such as to produce accelerations whose sum is zero, and then infer that the sum of the forces must be zero also, because forces are proportional to the accelerations which they produce on the same body. Or otherwise, as we have already put it (p. 250), if a body is visibly moving with uniform velocity, but is visibly also acted on by forces, we can only conclude that each force is producing its own acceleration, but that the forces are such that their whole accelerations exactly cancel each other. We should infer that if in such a case we took away any one of the forces, the body would at once begin to move faster or slower at a rate exactly corresponding to the now unbalanced part of the acceleration naturally due to all the remaining forces.

§ 37.—STATIC EQUILIBRIUM — GENERAL PROPOSITIONS.

The ordinary static problems connected with machinery are of two kinds: (i) the determination of the forces or moments acting on a body on the assumption that they must be such as to bring it into static equilibrium; and (ii) the finding of whether or not a body is in static equilibrium under the action of certain given forces. The first class of problems is far the most important, and we shall confine ourselves chiefly to it; the second class will not afterwards present any difficulty.

In practice the problem very generally takes this form:—Given that a mechanism is in static equilibrium, and that a certain external force or set of forces is acting upon it, to find what force in a given direction, and acting on a given link, is required to balance the given force or forces. The problem is simplest when all the forces are acting on the same link of the chain, but in practical work it often happens that given forces act on several links, and the force to be found acts on quite another. This is of course a more complex question, but we shall find that it can very readily be solved by an extension of the same methods which we shall use in the simpler cases.

The general conditions for the static equilibrium of a body having plane motion have been stated in § 36. They may now be put into more extended form in view of the problems just mentioned, and this is done in the following propositions:—

(i) **The sum of the external forces[1] may be zero.** This occurs very seldom, as it involves the very special case

[1] For definition of "external forces" see p. 259.

that the total action of all the given forces upon the link on which the unknown force acts should be in direction and position exactly opposite to that force. The case need not be separately considered, as it is most conveniently treated under one of the following :—

(ii) **The sum of all the external forces must pass through the virtual centre.** This is simply an inversion of the proposition on p. 264, and is the form of the proposition which will be found most generally useful.

(iii) **The moment** (p. 232) **of the sum of the external forces about the virtual centre must be zero,** or, what is the same thing, **the sum of the moment of all the external forces about the virtual centre must be zero.** This follows at once from (ii), for if the sum of the external forces is itself a force passing through the virtual centre, it cannot have any moment about that point. This form of the proposition is specially useful in enabling us to deal with couples, but is convenient in many other cases. If the sum of the external forces is zero, their moment (that is, the moment of their sum) is zero about every point in the plane, and not only about the virtual centre. But if the sum has any finite value it must have some moment about all points except those lying in its own direction line, and of these points the virtual centre must always be one.

It follows from (iii) that the effect of a force upon a body whose motion is one of rotation about a point at a finite distance, is not simply proportional to the magnitude of the force, but to its **moment**—the product of its magnitude and radius, or perpendicular distance from the virtual centre. A given force will produce the same linear acceleration in a body of given mass, whether the motion of the body be a simple translation or a rotation, *if only its* *radius is equal to the radius of inertia of the body.* In any other case the acceleration produced is not proportional to the actual force itself, but to its equivalent at the radius of inertia. If we write k for this radius and r for the radius of any force f, then the equivalent force at the radius k will be $f_o = f\frac{r}{k}$, for we have just seen that for two forces to be equivalent, or to cause the same acceleration in a body, their moments about the virtual centre must be equal, or $f_o k = fr$, of which equation the one given above is only another form.

The algebraical forms of the propositions given above are as follows :—

(i) If f_1, f_2, f_3, &c., be the forces, then

$$f_1 + f_2 + f_3 + \dots \quad = 0 \qquad (1)$$

or more shortly

$$\Sigma f = 0 \qquad (2)$$

The sum (Σf) is not the mere arithmetical sum of the quantities, but their geometrical sum, taking their directions and positions all into account. If they are all parallel each one must be supposed to be *intrinsically* positive or negative, so that the sum would be

$$(\pm f_1) + (\pm f_2) + (\pm f_3) + \dots = 0 \qquad (3)$$

(ii) and (iii). If f_1, f_2, f_3, &c., be again the forces, and r_1, r_2, r_3, &c., their virtual radii, while r is the radius of their sum,

$$r\Sigma f = 0, \qquad (4)$$

or

$$f_1 r_1 + f_2 r_2 + f_3 r_3 + \dots \quad = 0 \qquad (5)$$

Here the products or moments $f_1 r_1$, $f_2 r_2$, &c., are intrinsically negative or positive, according as the moment