CHAPTER VII.

DYNAMICS OF MECHANISM.

§ 23. LINEAR AND ANGULAR VELOCITY.

We have now completed our examination of the nature of mechanisms, as well as of a series of the principal kinematic problems connected with them. So far we have been able to work by methods which are in reality purely geometrical, and have not found it necessary to introduce questions of force or mass at all except to show that they might—for that part of our work—be put upon one side. We now leave the Kinematics for the Dynamics of Mechanism, and come to problems which involve directly questions connected with forces and the balancing of forces. Before we are in a position to deal with these problems it will be necessary to give some further attention to the meanings and relations of the ideas which are involved in them, velocity, acceleration, force, mass, and so on.

We have hitherto looked at motion merely as change of position, and in reference to velocity we have only noticed the relative (and not the actual) velocities of different points in the same body or mechanism at the same time. We have now to deal with problems which involve the determination of actual velocities and the velocities of the same point at different instants, problems which cannot be understood or solved without continual reference to the forces causing motion or change of motion.

We have distinguished (§ 15) between linear velocity and angular velocity. The former has to do with the distance, measured in ordinary length units, as feet, traversed by a point in a given time, while the latter is measured by the angle swept through by the radius of the point in a given time. Although something has already (in § 15) been said about the relation between linear and angular velocities, it is necessary to look at it here somewhat more in detail.

We have already seen (§ 7) that every body which has plane motion must be rotating about a point[1] in the plane. If this point be infinitely distant (like the virtual centre of a sliding pair), all the points in the body are moving at every instant in parallel straight lines and with the same velocity. In this motion of translation (which is thus merely a special case of motion of rotation) the velocity of the body is fully known if that of any one of its points be known. *The rate of motion of any point in its given direction of motion is the linear velocity of the point*, and in this case, to which we shall in the first instance confine ourselves, the linear velocity of any point is also the linear velocity of the whole body.

When we say that a body has a (linear) velocity 5, we mean that it moves at a rate which, if continued unchanged for a unit of time, would carry it through five units of distance in the given direction. The units of time and distance are commonly seconds and feet respectively, so that in the case supposed we should mean that the rate of motion

[1] The body, of course, rotates about an *axis*, but we have seen that the point which is the intersection of this axis with any plane parallel to the plane of motion, may be taken instead of the axis if we take instead of the body its section by or projection on the plane. See § 9.

of the point was such as would carry it through five feet in a second if it continued unchanged for that period.

But from the mere statement that a body has a linear velocity 5 *at a given instant*, we cannot infer that it possesses that velocity during any length of time, that, for instance, it will actually move five feet in a second, or still less that it will move 300 feet in a minute. We know only that at one particular instant it was moving at a rate which, if continued without change for a second, would have carried it through five feet in that time. This is not in any way inconsistent with its actual movement in the second being three feet or twenty feet instead of five, for its velocity, or rate of motion, may change altogether before the second is finished.

It has to be particularly remembered also that such a numerical value as that just given refers only to the *magnitude* of the velocity, and does not give any information about its *direction*, which, as we shall see, is in many cases equally important.

The velocity of a body is thus what may be called an "instantaneous" quantity. *At a certain instant* the body is moving at such and such a rate. The fact that this rate was quite different the instant before, and will be again quite different the instant after, does not in the least affect the matter. When, therefore, we represent the (instantaneous) velocity of a body by a line A A_1, we do not mean that the body actually moves from A to A_1 in a second, but only that when it is at A it is moving at a rate and in a direction which would bring it to A_1 in a second if only rate and direction continued unaltered for so long.

It happens, however, that we have often to concern ourselves with the *mean* velocity which a body has during a certain interval of time, that is to say, the mean of the velocities which it has at each successive instant throughout that time, instead of its velocity at one instant only. But a velocity over any interval of time may be *uniform* or *varying*. In the former case the velocity is the same at every instant, and the mean velocity during the whole time is one and the same with the instantaneous velocity at any instant whatever during the time. If a body have a uniform velocity v_0 and pass through a distance s in t seconds the relation between the three quantities is simply

$$s = v_0 t.$$

This relation is equally true if v_0 be the *mean* velocity of a body which has had varying velocity during the time t; when, that is, its velocities at different instants during the time have been different. But in that case the *actual* velocity might be v_0 at perhaps only one instant during the whole time, differing from it more or less at all other instants.

In the case of a varying velocity the rate of variation may itself (as we shall see in § 24) be uniform or varying. In the former case the mean velocity is the arithmetical mean between v_1, the initial velocity, and v_2, the final velocity,[1] or

$$v_0 = \frac{v_1 + v_2}{2} \qquad s = \frac{(v_1 + v_2)t}{2}$$

In such a case the mean velocity is therefore very easily found, and the actual velocity at any instant scarcely less easily. Fig. 92 (on p. 198) is a velocity diagram for such a case, where $v_1 = 1{\cdot}5$ ft. per second and $v_2 = 5$ ft. per second. v_0 is therefore $3{\cdot}25$ feet per second, which is the actual

[1] If the sense of v_2 is opposite to that of v_1 it must have the minus sign prefixed to it, and $v_0 = \frac{v_1 + (-v_2)}{2} = \frac{v_1 - v_2}{2}$

velocity at the end of half the time interval, or two and a-half seconds from the start. v_0 is of course the mean height of the line whose ordinates represent the velocities.

But in the case of a body moving with some irregularly varying velocity, such as that shown by the diagram (Fig. 90) on p. 194, the mean velocity can only be found approximately by taking the mean of the actual velocities at a sufficient number of different points. The arithmetical mean of the initial and final velocities may, in such a case, differ to any extent from the real mean, and could not be substituted for it.

It is very important in what follows that the distinction which we have just enforced between the velocity of a body at a given instant and the mean of its velocities at a number of successive instants, should be kept in mind, a distinction which applies equally to angular and to linear velocities.

The linear velocity of a body, as a quantity having magnitude, sense, and direction, is a "directed quantity," or vector, which has been our justification for the representation of velocities by lines having just those properties, and which justifies us in applying all the graphic rules of vector addition, &c. to lines representing linear velocities.

We have now to look at the case where a body (plane motion being always presupposed) is *turning about a point at a finite distance*, so that its motion is a simple *rotation*. Here, as we have seen, the linear velocity of every point is proportional to its radius, so that all points not having the same radius have different linear velocities. But although the points of a rotating body have so different linear velocities, yet as long as the form of the body is not itself undergoing change, all points in it move through the same angle in the same time. Otherwise, as we said in § 7, different parts of it must have had different motions, and this is impossible as long as the body remains rigid. The fact that all the points of a rotating body move through the same angle in the same time is expressed by saying that every point in it has the same angular velocity.

Just as either a foot, a yard, or a mile might be used as a unit for linear motion, so several different units might be used for angular motion—a revolution,[1] for instance, or a degree. There are practical conveniences, however, in taking for the unit of angular motion the angle subtended by an arc whose length is equal to its radius, which is $\left(\frac{360}{2\pi}\right)^{\circ}$ or about 57·3°. As the circumference of a circle of radius r is $2\pi r$, the number of units equal to r in one complete revolution is $\frac{2\pi r}{r} = 2\pi$, which is numerically equal to the distance moved through in one revolution by a point at unit radius. Further, if the body make n revolutions per second, it moves through $2\pi n$ angular units per second, which is again numerically equal to the number of feet traversed per second by a point at unit radius.

The number $2\pi n$ is called the angular velocity of the body, an expression which may be understood to mean either the rate at which the whole body is turning about its axis, expressed in angular units per second, or the rate at which any point in it having a radius equal to unity is moving, expressed in feet per second. The velocity in feet per second of any point in the body whose radius is r feet is obtained by multiplying the angular velocity of the body by the radius of the point, and is therefore equal to $2\pi n r$.

Where a body has a motion of translation only, it is often

[1] In cases where the revolution is used as the unit of angular motion the time unit is most commonly a minute instead of a second.

sufficient for us to take the velocity of *any* point as representing that of the whole body, just as if the whole body were concentrated at that point. But where the body is in rotation about a point at a finite distance, and in all problems which involve the action of forces on the body, and consequently involve consideration of its mass, we may suppose the whole mass to be concentrated only at any point among those which have one particular radius. This radius we may call the **radius of inertia** of the body,[1] and any point having this radius may be called a **centre of inertia** of the body. We cannot take its existence for granted without proof, and the proof will be found in § 32. **Its position is such that if the whole mass of the body were concentrated there in one small particle,[2] the action of any forces on that particle would be the same as their action on the whole actual body.**

The radius of inertia is not equal to the virtual radius of the centre of gravity, and indeed becomes widely different from it when the virtual radius is small relatively to the dimensions of the body.

If, therefore, R be the radius of inertia, in feet, of a body revolving about a given point (whether a virtual or permanent centre) with an angular velocity $2\pi n$, the body may be represented by a particle of equal mass to itself having a linear velocity $2\pi n R$ in a given direction, or normal to a given radius.

The linear velocity of any point in a rotating body is

[1] The terms "radius of gyration" or "radius of oscillation," which are sometimes used for it, are, unfortunately, very inconvenient.

[2] The "particle" is supposed to be indefinitely small, so that its size may be entirely negligible. In the case of a body rotating about its own centre of gravity, a thin cylinder or ring takes the place of this ideal particle.

thus a **moment**, and is numerically equal to the product of the angular velocity possessed in common by all the points of the body and the radius of the particular point in question.[1] The linear velocity of a point in a rotating body may therefore be represented by an *area*. It will be numerically equal to twice the area of the triangle whose base is the radius of the point and whose height is the angular velocity of the body.

Thus in Fig. 85 let a body be turning about O with an angular velocity v_a. The linear velocities of the points

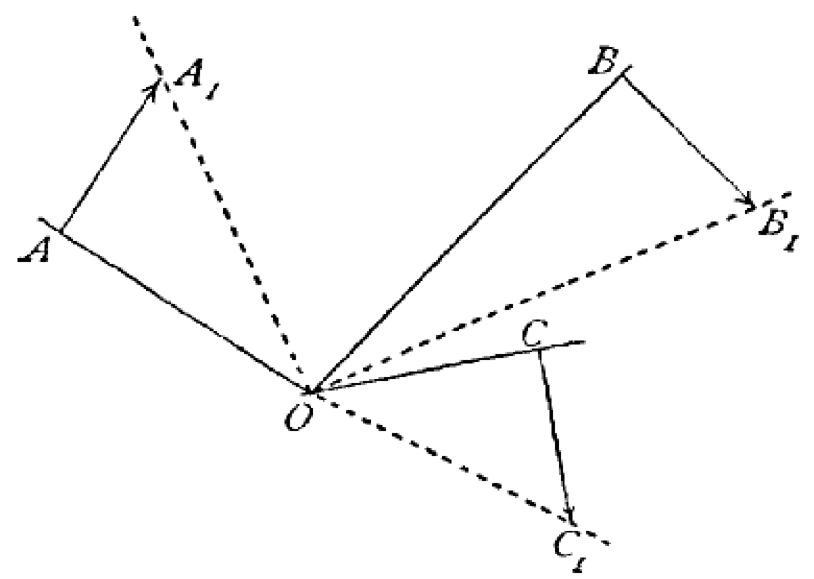

FIG. 85.

A, B, and C are proportional to the areas of the triangles AA^1O, BB^1O, and CC^1O, if AA^1, BB^1, and CC^1 are each made equal to v_a, and set off at right angles to their respective radii. The numerical value of the linear velocity of each point is obtained by doubling the area of the triangle in each case.[2]

In general the different points of a rotating body are

[1] It is presupposed that the units of linear and angular velocities are those stated above.

[2] If we applied this to the case of translation we should have the radius of every point infinitely great and the motion of the body measured in angular units infinitely small. The linear velocity would, therefore, come out in the form $\infty \times 0$, which cannot be further utilised directly.

moving in different directions, only those lying on the same radius having the same direction, but every point (§ 7) is moving at right angles to its own virtual radius.

§ 24. LINEAR VELOCITY—TANGENTIAL ACCELERATION.

So far as we know, a body which is at rest will remain always at rest, a body which is in motion will move for ever in the same direction with unchanged velocity, unless some extraneous cause alter the condition of rest or of uniform motion. Any such change is called an **acceleration**, and the "cause" producing acceleration is called **force**. It is necessary that the meaning and relations of both these expressions should be examined in some detail, and in the present and next following sections we shall consider the former of them.

A velocity [1] has magnitude, sense, and direction. **Any or all of these may undergo change, and any such change is called an acceleration.** But a change in sense is really only a change in magnitude. If, for instance, a body be moving with a velocity of 10 in a given direction and sense, and its velocity be changed to 5 in the same direction but in the opposite sense, we can say that its velocity has been changed from + 10 to − 5, and therefore the whole change is − 15, and can be entirely measured as a change of magnitude. We may therefore say that acceleration must be of one or other (or both) of two kinds, one affecting the magnitude and the other the direction of a

[1] See also §§ 14 and 15.

velocity. The first we shall call **tangential acceleration,** and the second **radial acceleration.** We shall in this section consider only the way in which changes in the magnitudes of velocities, or tangential accelerations, are related to each other and measured.

If the velocity of a body change from five feet per second to eleven, the magnitude of its velocity has been increased by six feet per second, that is, it has received a certain tangential acceleration. It is very important to notice, however, that we do not therefore say that it has received an acceleration of six feet per second, any more than we would say that the original velocity of the body was five feet. A foot is a unit of distance—a foot-per-second is the unit of velocity; and for acceleration our unit must be not a foot-per-second, but a *foot-per-second of velocity added in one second*, or more shortly a **foot-second per second.** A finite increase of velocity must have occupied some finite interval of time, say one second, or three. But there is just as much difference between an increase of velocity which occupies only one second and one which is spread over three, that there is between the traversing of a certain distance in one second or in three. In the latter case the *velocity* is in the one instance three times as great as in the other—in the former case the *acceleration* is in one instance three times as great as in the other.

We have seen in the last section that we may have either to do with the instantaneous velocity which a body actually possesses at a given instant or with the mean of its velocities during a certain succession of instants. We have now exactly the same distinction to make in the case of accelerations. The acceleration which a body is undergoing at a given instant is *the rate at which its velocity is changing at that instant,* measured in foot-seconds of additional velocity (positive or negative) per second. It does not follow[1] because a body has at a given instant an acceleration of six foot-seconds per second that therefore it will actually in any one second receive this additional velocity. All that we can say about it is that *if* the rate of change of velocity continued unaltered for a whole second the amount of the change would be six feet-per-second.

It frequently happens that our problems are connected not so much with the actual acceleration of a body at any given instant, as with the average value of its accelerations at each instant over some finite time-interval. In such a case we find the total change of velocity which has occurred,—that is the difference between the initial and final velocities,—and divide by the time in seconds to obtain the *mean acceleration during the time*.

The acceleration thus found is not necessarily the actual mean acceleration. It is the acceleration which, if it had acted uniformly for the given time, would have produced the given change of velocity in that time. But if the actual acceleration, as is most often the case, has been varying, it would at most instants differ from the mean acceleration thus found, and it might or might not be a part of our problem to find out at what instants the two values agreed.

The distinction between the actual acceleration at a given instant, and the mean acceleration over a given time, must be kept in mind as clearly as the analogous distinction (p. 163) between instantaneous and mean velocity.

In order, then, that we may measure the real change taking place in the velocity of an accelerated body we must know the rate at which the change is taking place, and our unit for measuring this rate of change, for which "acceleration" is only another name, is one foot-per-second of velocity

[1] See the similar case of velocities on p. 163.

added in one second—or one foot-second per second. The number of units of velocity which would be gained in a unit of time if the change continued uniformly for that time, measures the acceleration of the body, or its rate of change of velocity.

It cannot be too distinctly remembered that we cannot speak of an acceleration of so many feet-per-second. It is unfortunate that we have no short expression to stand for a unit of velocity, so that we are compelied to adopt the somewhat cumbrous phrase already used. If a foot-per-second were called (as Dr. Lodge suggests) a "speed," then the unit of acceleration might be said to be *one speed-per-second*. As it is, a foot-second per second seems the shortest available expression which we can use for it. When, therefore, we say that a body receives an acceleration of 10, we mean that it receives additional velocity at a rate which, if it remained unaltered for one second, would amount to ten feet per second in that time.

When the acceleration of a body is the same at successive instants it is said to be *uniform*, in all other cases it is *un-uniform*, or *varying*. If a body has a uniform acceleration over a certain period of time, its mean acceleration during that period is equal to its acceleration at each and every instant of the period. It is in such a case the same to us whether we have to do with the acceleration at one instant or the mean of the acceleration at many successive instants, for at every instant the acceleration is the same. As this case is so much simpler than that of varying acceleration we shall consider it by itself first, only premising that in a majority of the cases occurring in engineering problems the acceleration is varying, and that the assumption of uniform acceleration in some such cases may lead to serious error, if indeed it does not make the problems altogether meaningless.