CHAPTER V.

RELATIVE VELOCITIES IN MECHANISMS.

§ 14. RELATIVE LINEAR VELOCITIES

WE have so far considered motion only as change of position, entirely without reference to the time occupied by the change, that is, to the velocity of the different points of the body while moving; and we have seen that there are many kinematic problems which can be treated entirely without consideration of velocity. Connected with velocity, however, there are two distinct sets of problems which we have to examine, and one of these we can now take up. The absolute velocity of any point in a machine, as well as the changes in that velocity, depend, as we shall see presently, upon the forces acting upon the different parts of the machine. With these we have not at present anything to do. But the *relative velocities* of different points in the machine at any given instant can be determined by purely geometric considerations, so that we have already the means of dealing with them. We have seen that at each instant every body[1] in a machine or mechanism is virtually turning about some particular point, and have seen, further, how to find that point. **Every link of the machine, therefore, is simply in the condition of a wheel turning**

[1] Limiting ourselves to *plane motion;* see end of § 2.

about its axis, or a lever vibrating on its fulcrum, and this no matter how complex in appearance, or even in reality, the connection between the different parts of the machine may be. But in such a case it is obvious that the velocities of the different points must be simply proportional to their distance from the centre of rotation, that is proportional to their real, or virtual, radii or "leverage." The velocity of any one point being then known, the determination of the velocities of the others becomes a mere matter of finding the virtual centre and the distances of the various points from it. And even without knowing the *absolute* velocity of any point the same method gives us the *proportionate* velocities of all the points, quite independently of their absolute velocities. We must now look at this somewhat more in detail, especially in reference to *angular* as well as *linear* velocity.

When a body is turning about any fixed axis its motion is characterised by two conditions: (i.) the angular velocity of every point in it is equal, and (ii.) the linear velocities of its different points are proportional to their radial distances from the fixed axis, the linear velocities of points at equal distances from this axis being therefore equal. These conditions being characteristic of rotation simply, without reference to whether it occur for a short or a long time, are as entirely applicable to rotation about a virtual as about a permanent axis or centre. The difference is merely that in the former case the results obtained apply merely to one position of the body, while in the latter they apply equally to all its positions. We have seen that the motion of every link in a mechanism relatively to every other may at any instant be considered as a simple rotation about some point in that other. Hence it follows that at any instant every point in a link has the same angular velocity—that it describes, that is, equal angles in equal times.[1] It follows also that the linear velocities of different points in any link vary in direct proportion to the virtual radii of those points. Take Fig. 35 as an example, supposing *d* to be fixed, and

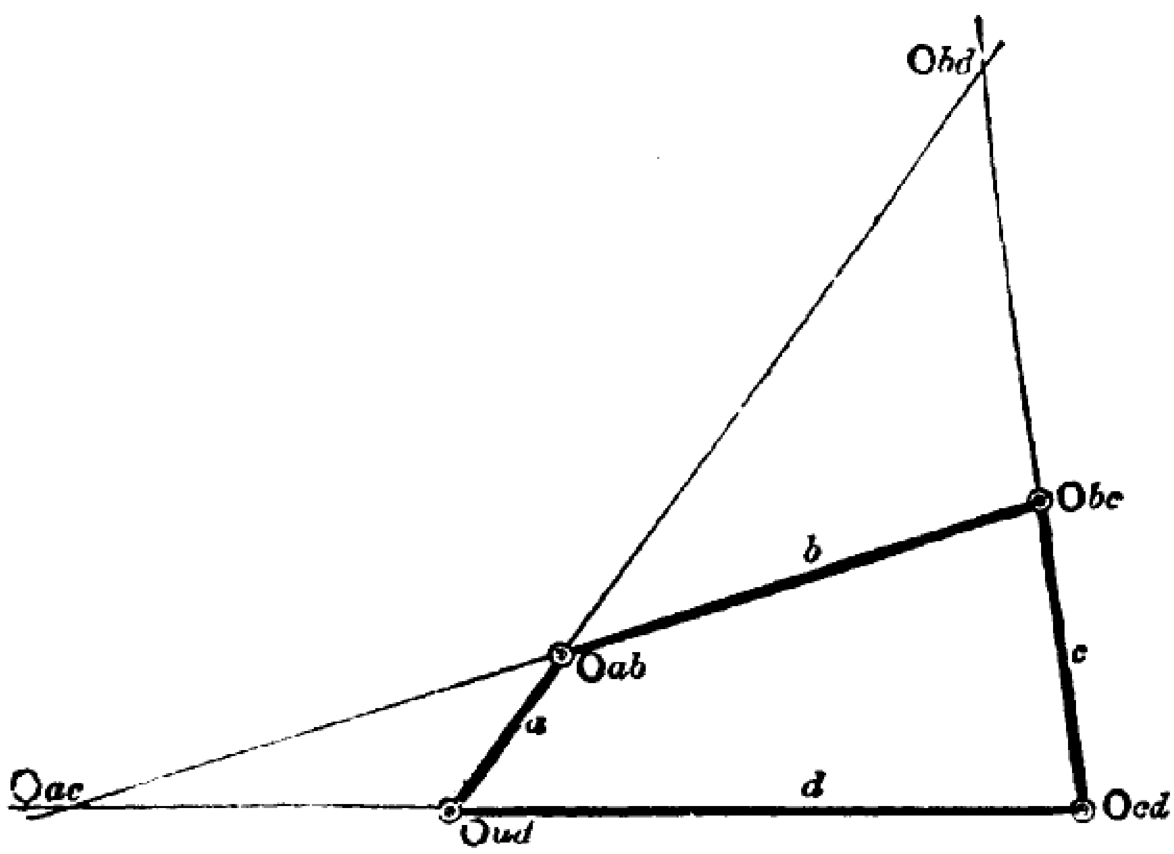

FIG. 35.

the motions of the other three links observed relatively to it. Every point in *a* is, at the instant, turning with the same angular velocity about O_{ad}, every point in *b* with the same angular velocity about O_{bd}, and every point in *c* with the same angular velocity about O_{cd}. Further, a point in *a* at any given distance from O_{ad} moves with just half the linear velocity of a point in *a* twice as far from O_{ad}, and with double the linear velocity of a point half its distance from O_{ad}, and similarly with the other links, whether the centres about which they are turning be permanent or virtual.

As we have seen, this makes it an extremely simple matter to find the velocity of all the points in any link if

[1] More fully that all the points *would* describe equal angles in equal times if they continued to move with the velocities which they have at the instant of observation.

only that of one point be known. Suppose, for instance, that the velocity of the point A_1 (Fig. 36) be given, to find that of A_2, both points belonging to the link *a*. Arithmetically it might be found by measuring $\overline{O_{ad}A_2}$ and $\overline{O_{ad}A_1}$, to any scale, and multiplying the given velocity by the ratio between them, *i.e.* by $\frac{\text{virt. rad. } A_2}{\text{virt. rad. } A_1}$. We shall

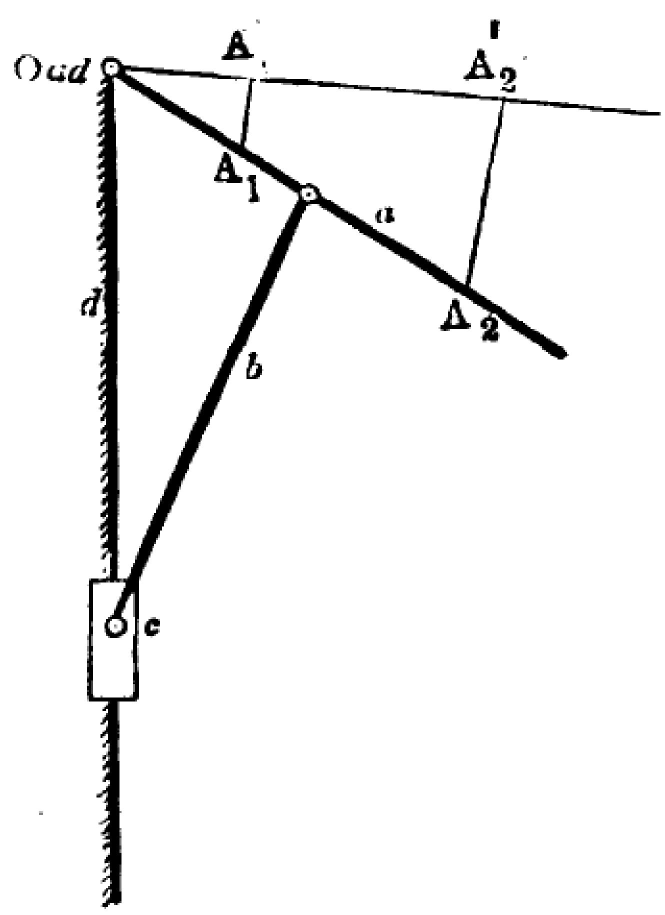

FIG. 36.

find it often more convenient, however, and it involves less measurement and no arithmetical multiplication, to solve the problem by a construction, as follows: Set off $A_1A'_1$ through the point A_1 in any convenient direction, to represent the given velocity of that point on any scale. Through O_{ad} draw a line through A'_1, and through A_2 a line parallel to $A_1A'_1$, calling the join of these lines A'_2;—the segment $A_2A'_2$ represents the velocity of A_2 on the same scale as that on which $A_1A'_1$ represents that of A_1. For the ratio

$$\frac{A_2A'_2}{A_1A'_1} = \frac{O_{ad}A_2}{O_{ad}A_1} = \frac{\text{virtual radius of } A_2}{\text{virtual radius of } A_1}.$$

Fig. 37 shows another construction, and one often more convenient than the foregoing, for solving a similar problem. Let $B_1 B_2$ be two points of a link b, and let $B_1 B'_1$ be the known velocity of B_1, to find that of B_2. Join both points to the virtual centre of b relatively to the fixed link, viz. O_{bd}.

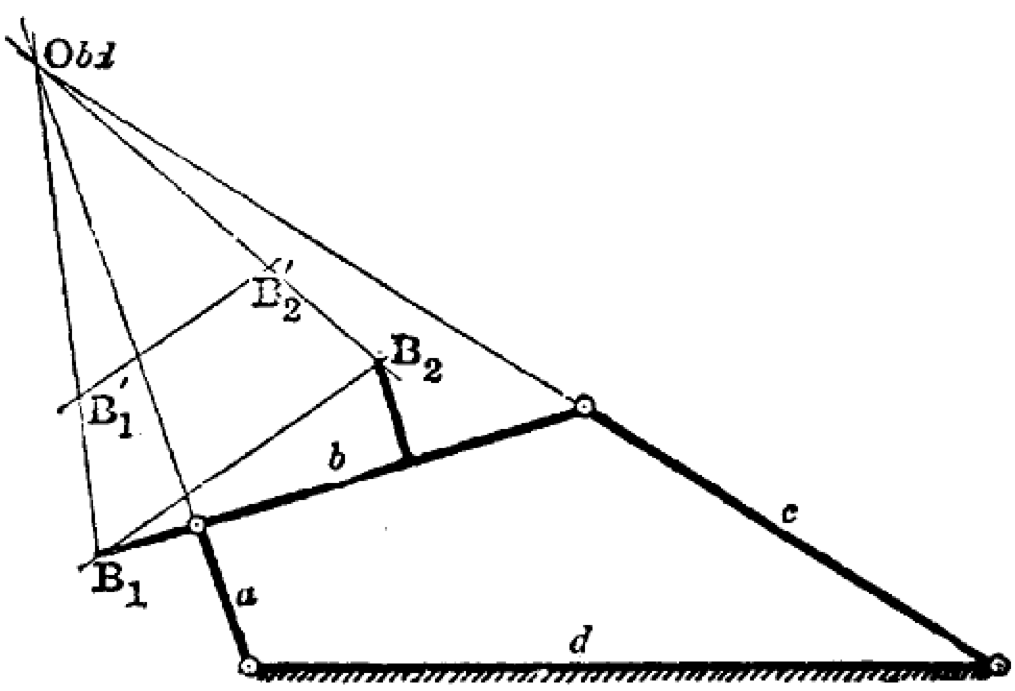

FIG. 37.

Join also B_1 and B_2, set off $B_1 B'_1$ along the radius of B_1 and draw $B'_1 B'_2$ parallel to $B_1 B_2$. Then $B_2 B'_2$ represents the linear velocity of the point B_2 on the same scale as that used in setting off $B_1 B'_1$. The proof is the same as before, simply that (by similarity of triangles)

$$\frac{B_1 B'_1}{B_2 B'_2} = \frac{O_{bd} B_1}{O_{bd} B_2} = \frac{\text{virtual radius } B_1}{\text{virtual radius } B_2}, \text{ as was required.}$$

It should be always most distinctly remembered that the bodies which are represented in our figures by straight links may be of any form whatever (see p. 67). We shall find that we have very often to do with points like B_2, Fig. 37, not lying at all on the axes of the bodies to which they belong. It should be noticed also that the line $A_1 A'_1$, &c., Fig. 36, were not set off in the direction of motion of A_1, &c., but in any direction that happened to be convenient.

We have compared the linear velocities of points of one and the same link,—but we can in just the same way compare the velocities of points in different links, or find the velocities of such points, if that of any one point be given. We do this by help of the theorem which we have already so often utilised, that the virtual centre of any link relatively to any other is a point common to both,—a point which has the same motion to whichever of the links we suppose it to belong. Let the velocity of a point A_1

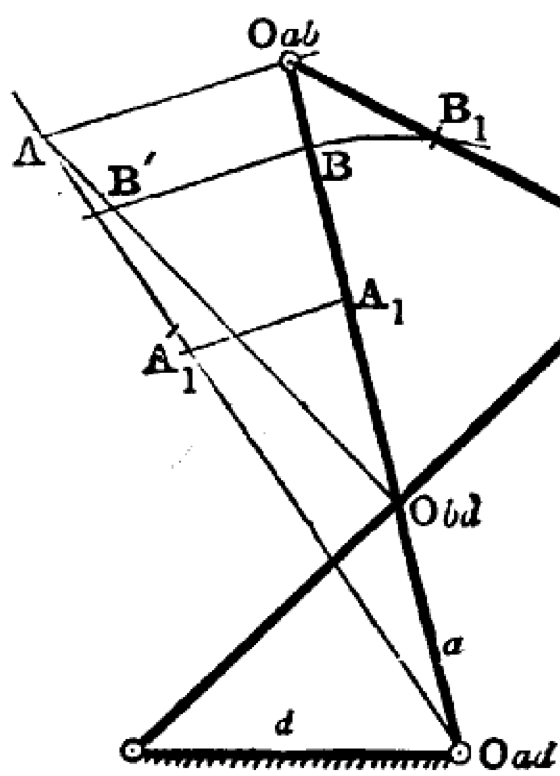

FIG. 38.

on the link a, for instance, be given;—to find from it the velocity of a point B_1 on the link b. The process is simply to find first the velocity of the common point of a and b as a point of a, and then treating it as a point in b to find from it the velocity of B_1. The necessary construction is shown in Fig. 38. $A_1A'_1$ is drawn to scale in any convenient direction for the velocity of A_1; by the former construction $O_{ab}A$ represents on the same scale the velocity of O_{ab} considered as a point of a. But this point has the same velocity as a point of b, so that by joining A to O_{bd} and carrying the radius of B_1 round to B, as in the figure, we get BB' for the velocity of B_1, to be measured on the same scale as before.

The construction applies equally to opposite as to adjacent links. To find, for instance, the velocity of the point C_1 in c, having given the velocity of A_1 in a as before, we should proceed as in Fig. 39, finding the velocity of

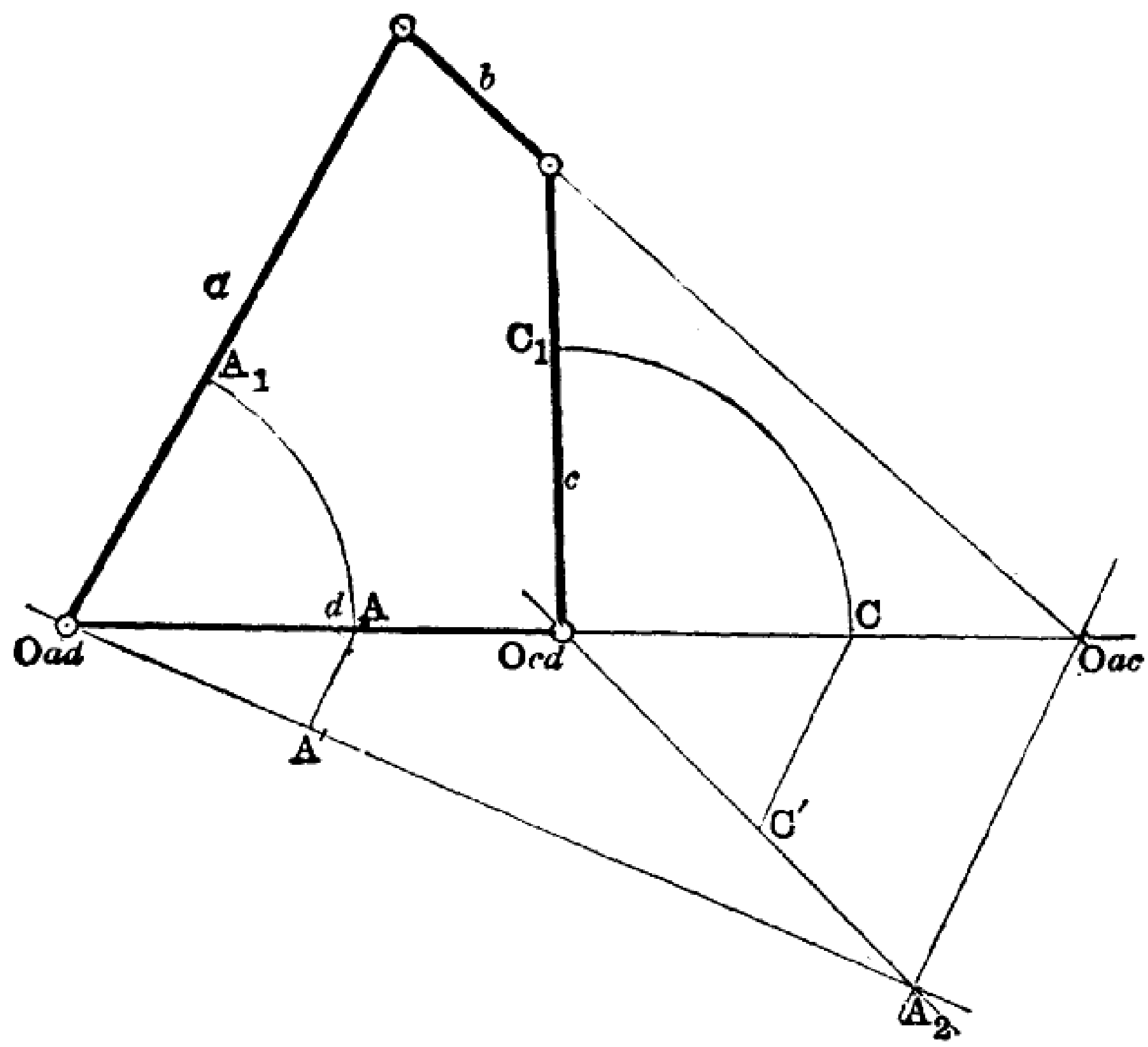

FIG. 39.

the point O_{ac} as a point in a, and then, treating it as a point of c, obtaining by its help the velocity of C_1. For convenience' sake we carry A_1 round to the line which is the axis of d, then setting off AA' as before, we obtain the line $O_{ac}A_2$ as the velocity of the point O_{ac}. Joining A_2 to O_{cd}, and carrying C_1 over to the axis of d (as we had previously done with A_1), we can at once draw CC' parallel to AA', and representing on the same scale the velocity of C_1.

Although the constructions of Figs. 37, 38, and 39 are very easy, both to work and to prove, they are not always the most convenient for practical purposes, mainly on account of the fact that it often happens that such points as O_{bd}, Fig. 37, or O_{ac}, Fig. 39, are at inconveniently great distances, often entirely beyond the limits of a drawing-board. There is never, however, any difficulty in dispensing with the actual construction of a line to such a point. Fig. 40, a very useful construction, shows how easily this

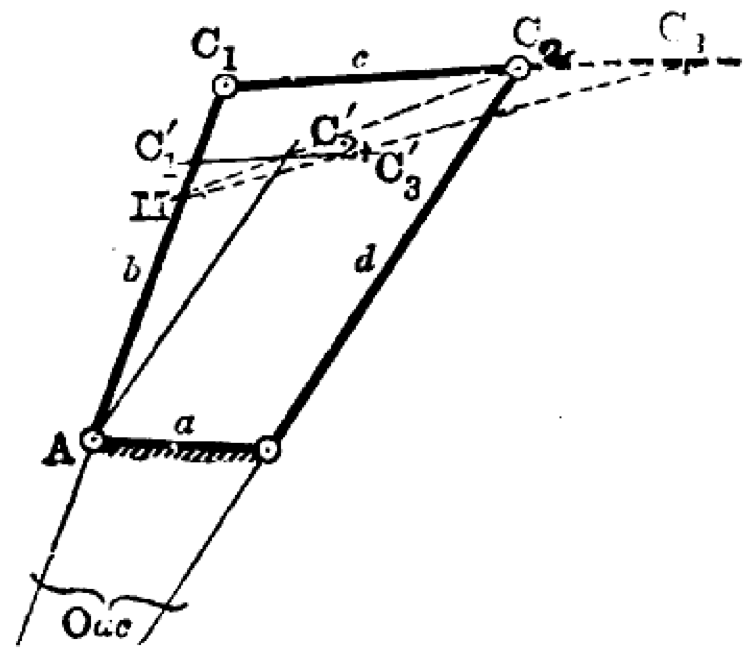

FIG. 40.

can be done. Let it be here required to find the velocity of the point C_2, that of C_1 being given. The point O_{ac} is inaccessible, but we have seen that it is sufficient for our purposes to have *any* two lines parallel to the virtual radii of C_1 and C_2, and not necessarily those radii themselves, with their (here) inaccessible join. If, therefore, we draw through O_{ab}, here the point A, a line parallel to d, we have at once such lines as we want in the most direct fashion. Setting off AC'_1 along b to represent the given velocity of C_1, and drawing $C'_1C'_2$ parallel to C_1C_2, then AC'_2 is the required velocity of C_2. For the triangles $O_{ac}C_1C_2$ and $AC'_1C'_2$ are similar, and hence $\frac{AC_1}{AC'_2} = \frac{\text{virtual radius } C_1}{\text{virtual radius } C_2}$, which is all that is required.

If the points C_1 and C_2 were both points the directions of whose virtual radii were known, as in the figure, it would be still simpler to proceed as in Fig. 41, where $C_1C'_1$ again represents the velocity of C_1, and $C'_1C'_2$ is drawn parallel to C_1C_2. Here $C_2C'_2$ obviously gives the velocity of C_2,—it is unnecessary to go through the proof. But we

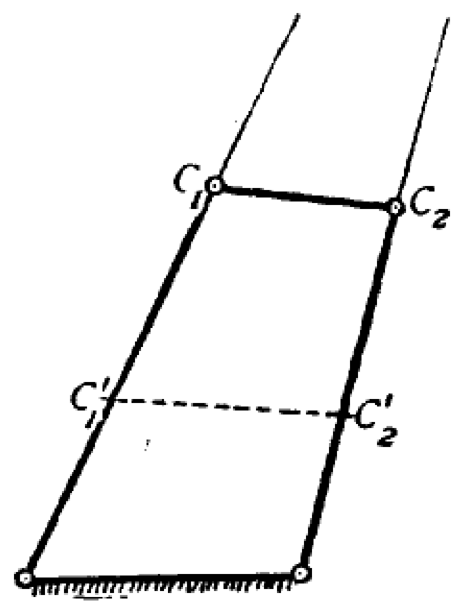

FIG. 41.

often have to deal with a point such as C_3 in Fig. 40, whose virtual radius is not a line in the mechanism, and is therefore not directly known, unless we can draw a line to the virtual centre, which is here supposed impossible. In this case we may conveniently proceed, as in Fig. 40, thus :— Find the join of the line $C_2C'_2$ with the link *b*, say *M*, and draw MC_3, which is cut by $C'_1C'_2$ in C'_3. The distance AC'_3 represents the velocity of C_3. The details of the proof may be left to the student.[1]

If in a case such as Fig. 39, the point O_{ac} be inaccessible, it can easily be dispensed with by the construction shown in Fig. 42. Here O_{cd} is joined to *S*, and a line drawn parallel

[1] It may be noticed that this construction gives one very easy way of drawing a line from a given point (as C_3), through an inaccessible point given as the join of two lines, *b* and *d*, on the paper. For of course a line through C_3 parallel to C'_3A would lie in the required direction.

to d in any convenient position, cutting the three lines which meet in S in the points 1, 2, and 3. The ratios of the distances between these points are exactly the same as between the three points O_{ad}, O_{cd}, and O_{ac}, which lie upon d, and it is only these ratios, not the points themselves, which we essentially require. The construction given in Fig. 39 may therefore be worked equally well by substituting 1 for O_{ad}, 2 for O_{cd}, and 3 for O_{ac}, and at the same time taking A' for A and C' for C,—in other respects working

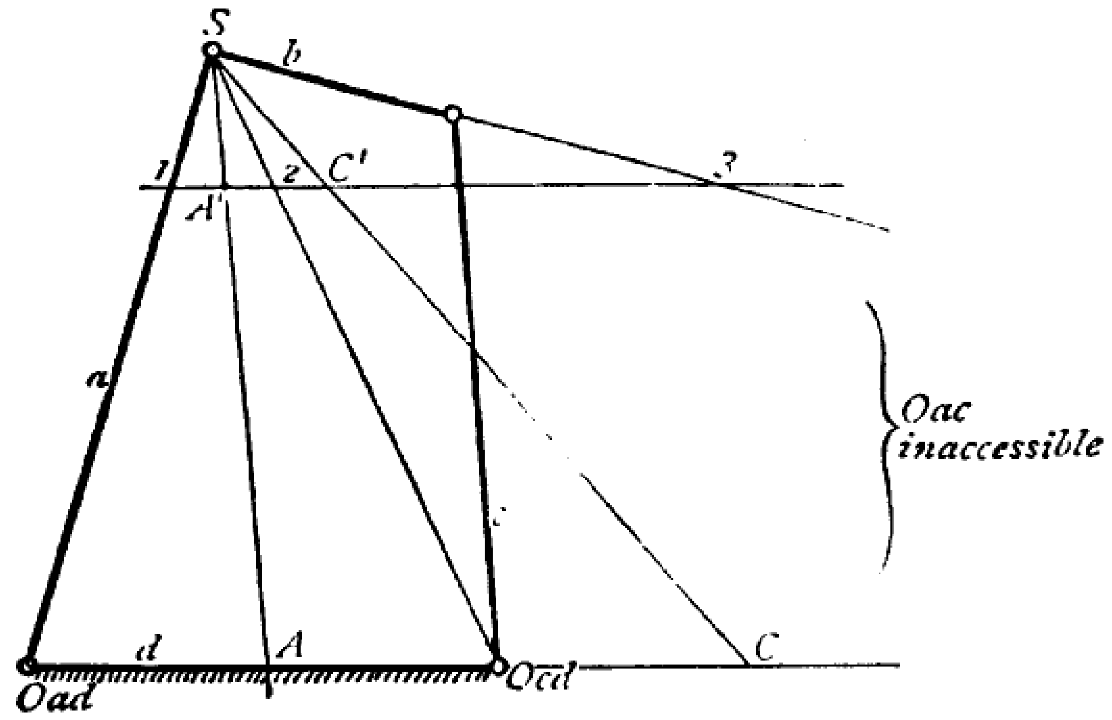

FIG. 42.

exactly as before. It can easily be seen that in effect this is nothing more than taking a different and more convenient *scale* for part of the construction. These constructions for getting over difficulties caused by inaccessible points are only examples of many that can be used, but are themselves quite generally useful, and are besides sufficient to indicate to the student the sort of procedure to be adopted in any particular case. Examples of other, more or less similar, constructions will be found further on, in connection with problems where they are required.

The whole matter which we have gone over in this section may now be summed up. Our problem has been: Given the linear velocity v_1 of any point A of a link a in a mechanism having plane motion, to find the simultaneous linear velocity v_2 of any point C of any other link c of the same mechanism, the fixed link being (say) d. Finding first the three virtual centres O_{ac} (which we may call O), O_{ad}, and O_{cd}, we have found that

$$\frac{\text{vel } C}{\text{vel } A} = \frac{v_2}{v_1} = \frac{OO_{ad}}{AO_{ad}} \times \frac{CO_{cd}}{OO_{cd}} = \frac{OO_{ad}}{OO_{cd}} \times \frac{CO_{cd}}{AO_{ad}}.$$

Put into words this is equivalent to saying that **the velocity of C is to that of A directly as the virtual radii of those two points relatively to the fixed link, and inversely as the virtual radius in c and in a of the common point (O_{ac}) of those bodies.**

If the two points belong to the same link, the ratio $\frac{OO_{ad}}{OO_{cd}}$ goes out, and we have simply that the velocities of the two points are proportional directly to their virtual radii. Here, however, one special case requires looking at. If both the points belonged to such a body as the link c in Fig. 36, their virtual radii,—no matter what their position in the body,—would always be equal. For the virtual centre of c relatively to d is a point at infinity, the distance of which from all points in our paper must be taken to be the same. Hence if the virtual centre of a body be at infinity, *i.e.* if it have only a motion of translation, all its points are moving with equal velocities. Exactly the same thing is true in reference to the link b in Fig. 43. In this mechanism, the **parallelogram** or **double-crank**,[1] opposite links are made equal, *i.e.* $b = d$, and $a = c$. Opposite links are therefore

[1] As to important properties of this mechanism see further, §§ 54 and 55.

always parallel, and their join is always at an infinite distance;—the points O_{bd} and O_{ac} are at infinity for all possible positions of the mechanism. Whichever link, therefore, is fixed, all the points of the opposite link are moving at any instant in the same direction and with the same velocity. The difference between the case of the link *c* in Fig. 36 and that of *b* in Fig. 43 is that the virtual centre of the former

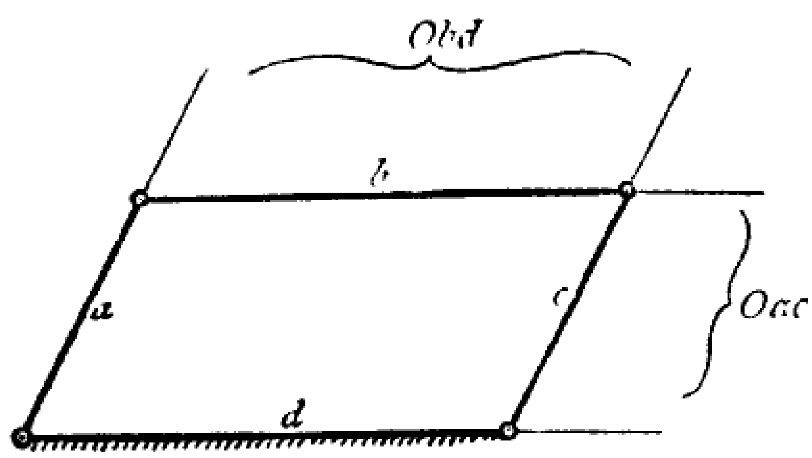

FIG. 43.

is a *permanent* centre, while that of the latter is only an *instantaneous* one. In the former case not only are all points moving in the same direction at any one instant, but this direction remains unchanged from instant to instant, whereas in Fig. 42 the direction of motion of *b* changes with every change in its position, although in any one position all its points are moving in the same direction. The difference is in essence precisely the same as that between the rotation of such links as *a* and *b* in Figs. 36 or 37. The motion of each link is at each instant a rotation about some one point. But in the case of *a* the rotation is always about the same point, in the case of *b* about a point which changes with every change in the position of the link.

§ 15. RELATIVE ANGULAR VELOCITIES.

Just as linear velocity may be expressed in different units,—as a velocity of a foot, a metre, or a mile per unit of time,—so angular velocity is a quantity measured by more than one standard.[1] The unit most commonly occurring in connection with engineering questions is *one revolution per unit of time*, the latter being generally a minute. Thus a shaft is said to have an angular velocity of 30 if it be turning at the instant at such a rate as would, if uniformly continued for one minute, cause it to make 30 complete turns in that time. To find the linear from the given angular velocity of the point in this case it is necessary simply to multiply the latter by the radius of the point and by 2π, that is, by the length of the circumference of the circle in which the point is moving. This assumes, of course, that the units of distance and of time are the same for both linear and angular velocities. For mathematical purposes the unit of angular velocity is generally taken as motion through an arc equal in length to its own radius in a unit of time. This arc subtends an angle of $\left(\frac{360}{2\pi}\right)$, or 57·3 degrees nearly, so that an angular velocity of 30 would represent, on this scale, a motion through $(30 \times 57{\cdot}3)$ degrees, or about 4·77 complete turns per unit of time. To convert angular into linear velocities on this scale the former have only to be multiplied by the radius. To convert angular velocities expressed in the former standard, therefore, to the latter, they must be divided by 2π, and *vice*

[1] The principal questions relating to linear and angular velocities are discussed in Chapter VII. What is said in the present section is not intended to do more than make clear the numerical relations of the units used as far as is necessary for the constructions given.

versâ, the time-unit being supposed the same in both cases. For general scientific purposes the second is the most convenient unit of time, but for many engineering problems the minute is preferable. For angular velocities expressed as number of revolutions, for instance, the minute is almost invariably made the time-unit.

There is obviously no more difficulty in solving problems connected with relative angular velocities than we have found in connection with relative linear velocities. It has only to be remembered, in addition to the characteristics of pure rotation already mentioned, that if two points of different bodies have the same radius, and have equal linear velocities, their angular velocities are also equal; and that otherwise (*i.e.* if the points have *unequal* linear velocities), their angular velocities are directly proportional to their linear velocities. If the two points have the same linear velocities but different radii, their angular velocities are inversely to their radii. In general, therefore, the angular velocities of two points in different bodies are proportional directly to their linear velocities and inversely to their radii. But as all points in a body must have, at each instant, the same angular velocity, we may say, even more generally, that **the angular velocities of any two bodies having plane motion are proportional directly to the linear velocities of any two of their points having the same radius, and inversely to the radii of any two of their points having the same linear velocity**, and in the general case to the ratio $\frac{\text{linear velocity}}{\text{radius}}$ for any two of their points whatever. We may put this down in symbols as follows:—calling a the linear velocity of any point A of a body α, and b that of any point B of another body β, the radii