CHAPTER IV.

VIRTUAL MOTION IN MECHANISMS.

§ 12. DETERMINATION OF THE VIRTUAL CENTRE IN MECHANISMS.

WE have seen that in order to determine the virtual centre about which a body is moving at any instant, it is necessary and sufficient to know the direction of the motion of two points in the body at that instant. We must now consider this more in detail, in order particularly to apply our knowledge to the solution of the problem in the case of mechanisms.

The path in which a point is moving in the plane may be supposed given, either by its equation or by its form actually traced out on paper. In the former case the direction of motion, or tangent to the curve, can be calculated, and in the latter case it can be drawn. We have to deal exclusively with the latter case in our work. There are few cases in which it is at all difficult to draw the actual path in which any point of a mechanism is moving, and to construct a tangent to this path at any point, and no cases at all, so far as we know, in which it is not greatly more convenient to do so than to calculate an equation to that path. Finding the direction of motion of a figure then means, for us, simply drawing the paths of two of its points and constructing tangents to them, or of course (if possible) constructing the tangents without drawing the point-paths themselves, which are not, in most cases, of any direct importance to us.

We require to know the direction of motion of two points in the body. *Any* two points will serve, provided their virtual radii be not coincident, in which case, of course, they would not determine the intersection which we require. The problem, therefore, resolves itself simply into a choice of points, a matter which we must here examine briefly

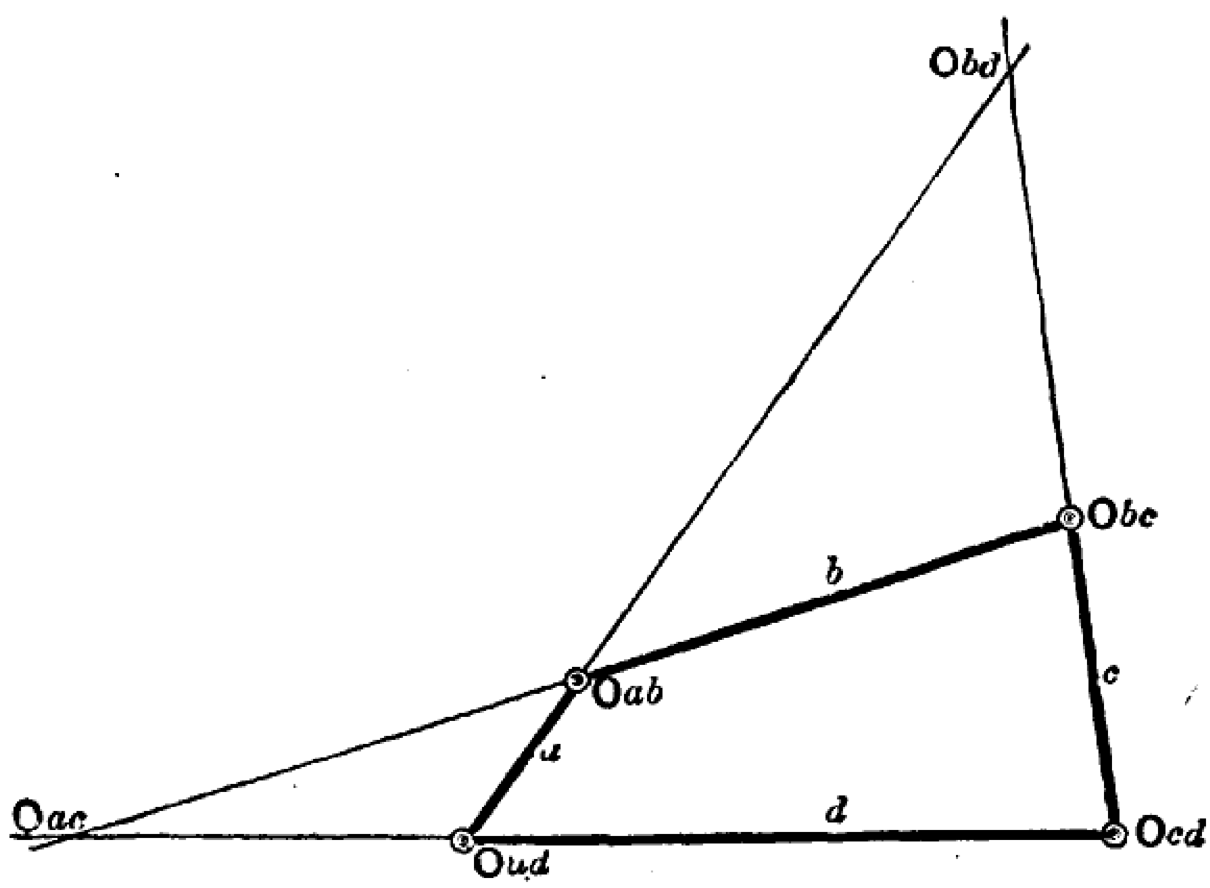

FIG. 29.

—we shall have numerous examples in succeeding chapters. To start with the simplest possible case, let it be required to find the virtual centre of every link relatively to every other in such a mechanism as Fig. 29, which consists of four links connected by four turning pairs. The axes of the pairs are all parallel, so that the links have only plane motion. Calling the links *a*, *b*, *c*, and *d*, we shall call their virtual centres O_{ab}, O_{bc}, etc., the suffixes denoting the links for which the particular point O is the virtual centre. The virtual centres of *adjacent* links are permanent centres, and are, as we have already seen, simply the centres of the pairs connecting them. Relatively to *d* for instance, every point in *a* moves always about the point O_{ad}, which is the centre point of the pair connecting *a* and *d*. By mere inspection therefore, we have at once the points O_{ab}, O_{bc}, O_{cd} and O_{da}, as the virtual (and permanent) centres of the four pairs of adjacent links. There are two other virtual centres in the mechanism, those for the two pairs of non-adjacent links; O_{ac} for the links *a* and *c*, and O_{bd} for the links *b* and *d*. We may take the latter first;—*b* is connected to *d*, and its motion constrained, by the links *a* and *c*, and we know the motion of every point in these two links relatively to *d*. But *b* has one point in common with *a*, viz. the point O_{ab} and also one point in common with *c*, the point O_{bc}. We know the motion of these points relatively to *d* as points of *a* and *c*, and of course they must have the same motion relatively to *d* as points of *b*, for they cannot have two different motions relatively to the same body at the same time. We have therefore at once the motion of two points in *b* relatively to *d*, which is all we require. O_{ab} is moving in a circle round O_{da}—without drawing its path then, or even constructing the tangent to it, we can at once draw its virtual radius, which is at right angles to the tangent, and which is simply the axis of the link *a*. In exactly the same way the axis of *c* is the virtual radius of O_{bc} and is at right angles to the direction in which it is moving. The virtual centre of *b* relatively to *d* is therefore at the join of these two axes, as shown by the fine lines in the figure. By the same reasoning it can be shown at once that the point O_{ac} is at the join of the axes of *b* and *d*.

Quite generally, therefore, in a chain such as Fig. 29, consisting of four links connected by four parallel turning pairs, **the virtual centre of either pair of opposite links is the join of the axes of the other pair; the virtual centre of any pair of adjacent links is the join of their own axes, and is a permanent centre.**

An inspection of Fig. 29 shows a rather remarkable regularity in the disposition of the virtual centres. The six centres lie in threes upon four lines, and the three centres on

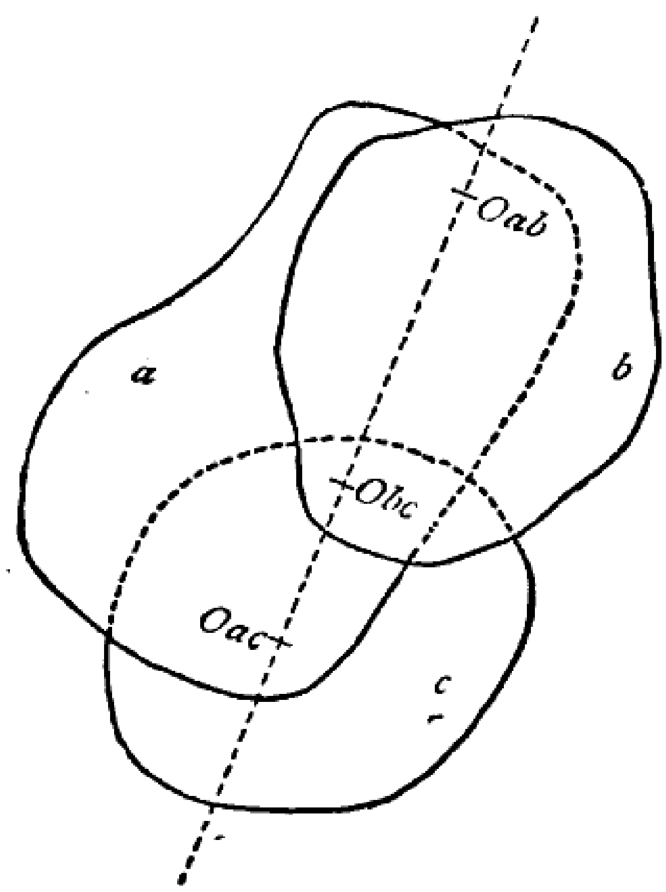

FIG. 30.

any one line are always those corresponding to three particular links out of the four. The links *a*, *b*, and *c*, for instance, give us the three virtual centres O_{ab}, O_{bc}, and O_{ac}, and these three points are in one line, here the axis of *b*. The links *b*, *c*, and *d*, similarly, give us the points O_{bc}, O_{cd}, and O_{bd}, and these again lie all on one line, here the axis of *c*. The question comes at once whether this is some mere coincidence, belonging to the very simple mechanism which we have chosen for illustration, or whether it represents some general law which we may apply in other cases. It is in fact quite general, and the proof is simple. Let a, b, and c (Fig. 30), be any three bodies whatever, having plane motion, and let O_{ab}, O_{bc}, and O_{ca} be the virtual centres for their motion. O_{ac} is a point both of a and of c; as the former it is moving about O_{ab} relatively to b, as the latter about O_{bc}. That is, its direction of motion as a point in a is at right angles to the line joining it to O_{ab}, and its direction of motion as a point in c is at right angles to the line joining it to O_{bc}. But it can have only one direction of motion relatively to b, whether it be treated as a point of a or of c, and as this direction is normal to both the lines just mentioned, they must either be parallel or coincident. They cannot be parallel, for they both pass though the same point O_{ac} on the paper—they must therefore coincide. The radius O_{ac} O_{ab} coincides with O_{ac} O_{bc}—the three points named therefore lie in one straight line. We might have started with O_{ab} or O_{bc} instead of O_{ac} and should always have come to precisely the same result, which may be summed up as follows; **If any three bodies a, b, and c, have plane motion, their three virtual centres O_{ab}, O_{bc}, and O_{ac} are three points upon one straight line.**[1]

We may now examine another simple chain, the one shown in Fig. 31, which is the same as one which we have already noticed. Using the same notation as before, we have again the points O_{ab}, O_{bc}, O_{cd}, and O_{ad}, the virtual centres of adjacent links, at once. Three of them are, as in the last case, the centres of turning pairs, and the fourth is the centre of the sliding pair $c\ d$, and therefore a point at infinity. All four, including, O_{cd} (see pp. 43 and 46),

[1] This proof, it may be noticed, is quite independent of the bodies being adjacent links in a mechanism, or indeed of their belonging to a mechanism at all; it applies to any constrained *plane* motion. For the corresponding theorem in spheric motion see § 63.

are permanent as well as instantaneous centres. The virtual centres of opposite links, O_{bd} and O_{ac}, are as easily found as in the last case, but the points which determine them are not, perhaps, quite so obvious. The link *b* has, as before, one point in common with each of the links *a* and *c*; we know the motion of every point in these links relatively to *d*, for we have found O_{ad} and O_{cd}, we therefore know the motion of two points in *b* relatively to *d*, these points being

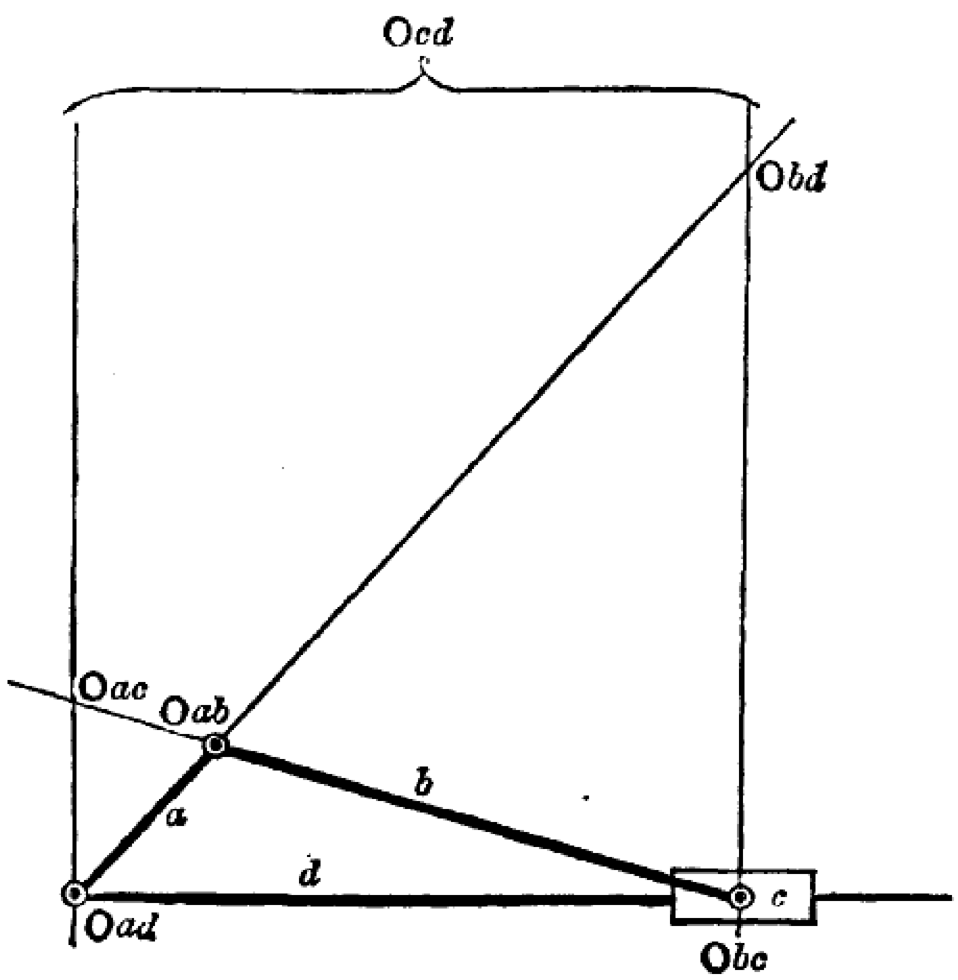

FIG. 31.

O_{ab} and O_{bc}. The virtual centre of *b* relatively to *d* is at the join of the virtual radii of these points, exactly as in the former case. The virtual radius of the point O_{ab} is the line joining that point to O_{ad}, or the axis of the link *a*. The virtual radius of O_{bc} is the line joining that point to O_{cd}, which is simply a line perpendicular to the axis of the sliding pair. The construction lines and the point O_{bd} at their join are shown in the figure.

By similar reasoning we get the point O_{ac}, the virtual centre of *a* relatively to *c*, but the reasoning is perhaps a little more difficult to follow. The link *c* has two points whose motion relatively to *a* we know, for it has one point in common with each of its adjacent links *b* and *d*, both of which are adjacent to *a*. O_{ac} must be, as before, the join of the virtual radii of these two points. The virtual radius of the one, O_{bc}, is the line joining it to O_{ab}, or simply the axis of the link *b*, and can at once be drawn. The virtual radius of the other, O_{cd}, is the line joining O_{ad} and O_{cd}. But O_{cd} is a point at an infinite distance, hence all lines passing through it are parallel on our paper (p. 43), so that to draw the line in question we have only to draw through O_{ad} a line parallel to the virtual radius of O_{bc} already constructed, (and therefore perpendicular to the axis of the sliding pair) and we have the line required, which gives us O_{ac} directly. The construction is shown on the figure.

By the help of the theorem about the virtual centres of three bodies which we proved above, this proof can be much shortened. From the fact that *a*, *b*, and *c* are three bodies having plane motion, we know that the point O_{ac} must lie on the line joining O_{ab} and O_{bc}, and similarly, considering the three bodies *a*, *c*, and *d*, we know that O_{ac} must lie on the line joining O_{ad} and O_{cd}. To find O_{ac} therefore, we have only to draw these lines to their join, which is just what we have done. Similar reasoning would, equally briefly, have given us the position of the point O_{bd}.

As these mechanisms are very important and will often be referred to, it may be well to use the name "lever-crank" for Fig. 29, and "slider-crank" for Fig. 31, the link *d* being supposed the fixed one in each case.

Fig. 32 shows a mechanism having a very close relationship to Fig. 31, but one in which it may appear at first sight to be a very much more difficult matter to determine the virtual centres. The difficulties which occur in this case are more apparent than real, but as they occur more or less frequently, it will be quite worth while working through the determinations in detail, and then trying to find out what the real differences are between the three mechanisms shown in Figs. 29, 31, and 32.

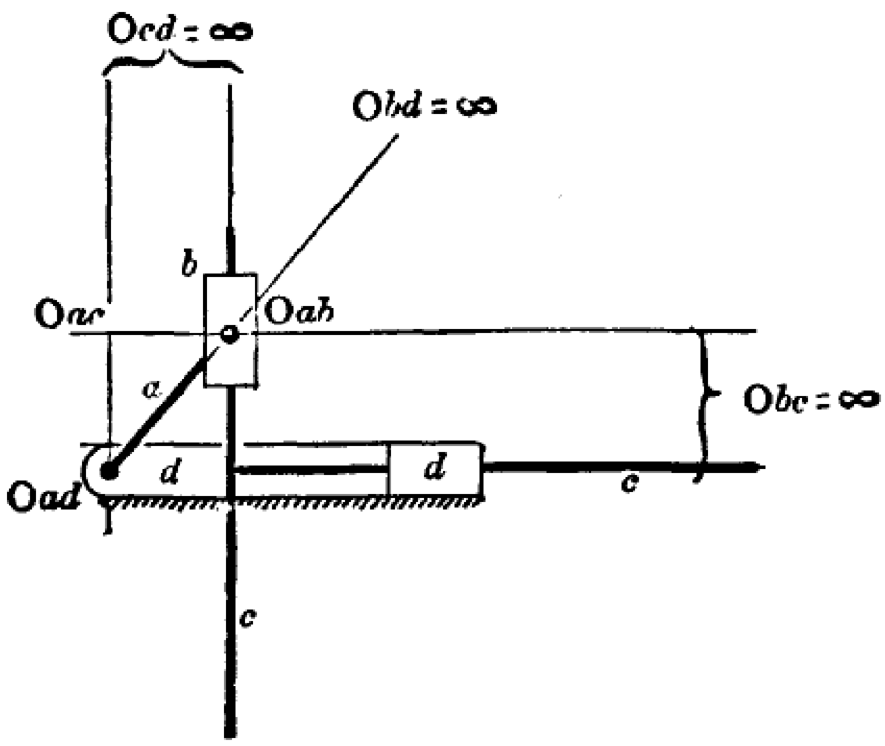

FIG. 32.

In Figs. 31 and 32 the links *a* and *d* are exactly the same, and also the pairing of *d* with *c*, and of *a* with *b*. But the link *c* now (Fig. 32) consists of two sliding elements instead of one sliding and one turning, while the link *b*, being paired to *c*, takes a form similar to that which *c* had before, viz., one turning and one sliding element. The virtual centres O_{ab}, O_{ad}, and O_{cd} remain exactly as before,—we may proceed to find the others. Take the point O_{bc} first: the link *b* simply slides upon *c*, their virtual centre must therefore be a point at infinity, and the direction of that point must be at right angles to the known direction of the sliding or translation of *b*. The position of O_{bc} must therefore be as shown in the figure. The point O_{ac} is equally easily found: it must lie upon the line containing O_{ad} and O_{cd}, and also upon the line containing O_{bc} and O_{ab}, and therefore must be at the join of these two lines at the point marked. We come lastly to the virtual centre O_{bd}. This must, in the first place, be on the line joining O_{ad} and O_{ab}, the axis of the crank. It must also lie upon the line which joins O_{cd} and O_{bc}. But *both* these points are at an infinite distance, where then *is* the line joining them? That we know no more than we know where the points themselves are (see p. 14). But reasoning from known properties of lines within our reach, we conclude that such a line must be *entirely* at infinity, *i.e.* that it must be treated as having *all* its points at an infinite distance. So therefore we say that the point O_{bd}, being a point upon a line which has already two points at infinity, must be itself necessarily at infinity also, and as its direction is already known, the point itself is completely fixed, as marked in Fig. 32. It follows, of course, that *b* should have only a motion of pure translation (p. 15) relatively to *d*, and this it can readily be proved to have.

A closer examination of Figs. 29, 31, and 32 will help to show some interesting relations between the mechanisms represented in them. In Fig. 29 we saw that all the six virtual centres lay in threes upon four straight lines, the axes of the four links. In Fig. 31 the six virtual centres also lie in threes upon four lines, but at first sight these do not appear to be the axes of the links, for the apparent axis of *d* (the lower horizontal line in the figure) is not one of them, while two of them (the two vertical lines) are not apparently axes of links at all. The discrepancy is only apparent. If we could construct a mechanism which, having *a* and *b* exactly as in Fig. 31, had *c* and *d* stretching away from their end-points to a point O_{cd} *at infinity*, and there connected by a pin, we should have a mechanism (see § 52) which might be called an ideal form of that shown in Fig. 31. Its links would have exactly the same relative motions—the point O_{bc} and all other points of the new link *c*, would move in straight lines parallel to the line *d* in the figure, because they would be turning about a point at an infinitely great distance, while its four links would be connected simply by four turning-pairs as in Fig. 29. And the four lines which would then form the axes of the mechanism are exactly those upon which the virtual centres do lie in threes. Those lines, therefore, may be considered to be the ideal axes of the links in the mechanism. But as we cannot construct a pin-joint at infinity we have to content ourselves with imitating the motions to be obtained from it by the use of a sliding pair. Kinematically, therefore, the mechanism of Fig. 31, the slider-crank, may be said to be the same as that of Fig. 29, with the links *c* and *d* made infinitely long.

By precisely similar reasoning, which it is unnecessary here to repeat at length, but which the student will find it a very useful exercise to write out in full, it can be shown that the mechanism of Fig. 32 is derived from the slider-crank by making the link *b* infinitely long. Of the four ideal axes only one (that of *a*) coincides with the actual axis of a link, two of the others are lines parallel to the arms of the link *c* and passing through the points O_{ad} and O_{ab} respectively, while the last is the line at infinity to which we have already referred. Fig. 32, therefore, represents the constructive form taken by a slider-crank with an infinitely long connecting-rod, a mechanism frequently referred to for practical purposes. In § 52 these points will be found more completely treated from a somewhat different point of view.

The use of the theorem that the virtual centres of three bodies having plane motion are three points upon a line, greatly shortens the proof in some cases of simple mechanisms, but such cases can generally, if not always, be proved without it. It is, however, indispensable when we have to find the virtual centres of links in compound chains. One example of this case will suffice for the purposes of the present section. We may take the compound chain of Fig. 33, which is similar to the one shown in Fig. 28. Here

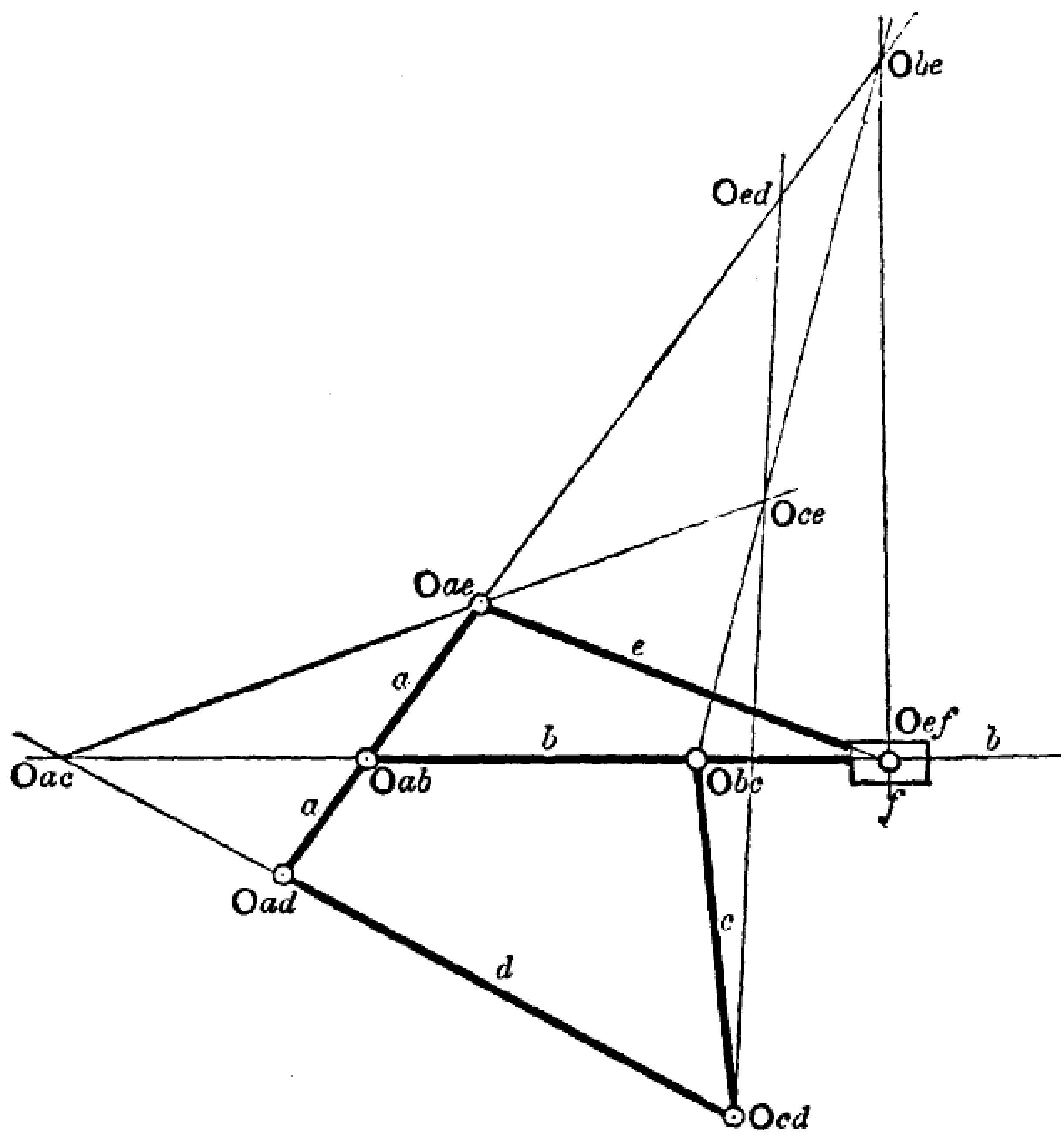

FIG. 33.

there are in all fifteen virtual centres lying in threes upon twenty lines, some of the latter, however, being coincident. The virtual centres of adjacent links can be marked at once, just as in a simple chain, and of the rest we may pick out three simply to illustrate the method of dealing with such cases. Let it be required to find the virtual centres of *c* relatively to *e*, of *a* relatively to *c*, and of *d* relatively to *e*. We see by inspection that we have already a line which contains O_{ab} and O_{bc} and which must therefore contain O_{ac}, one of our required points. But we have also a line containing O_{ad} and O_{dc}, upon which, again, O_{ac} must lie. The point O_{ac} must, as it is upon both these lines, be their join, which therefore can be marked at once. So far we have nothing different from a problem already solved (p. 71), the links *a*, *b*, *c*, and *d* forming by themselves a mechanism the same as that of Fig. 29. The position of the point O_{ac} is not affected in any way by the additional links *e* and *f*. The second required point O_{ce} must (as one of the three virtual centres of the three bodies, *a*, *c*, and *e*) be upon the line joining O_{ac} with O_{ae} which we have now the means of drawing. We have not, however, as in the last case, any other line containing it actually given by the mechanism itself, and must therefore proceed to find one. We have in our figure the point O_{bc}. This point must lie in one line with O_{ce} and O_{be}. But the latter can be easily found, for by the proof given in reference to the links *b* and *d* in the chain of Fig. 31, it must lie at the join of the axis of *a* with a line through O_{ef} normal to the axis of *b*. Drawing these two lines we get O_{be}, and joining this point to O_{bc} we have a line containing the required point O_{ce} which must therefore be at the join of this line with the one mentioned above;—its position is marked in the figure. By similar reasoning we can find the third required point O_{de}. The links *c*, *d*, and *e* being three bodies having plane motion, O_{de} must lie on the line joining O_{cd} and O_{ce}, and, for a similar reason, it must also lie upon the line joining O_{ae} and O_{ad}. Both these lines can be drawn, and their join is the required point O_{de}.

Similarly all the rest of the fifteen virtual centres belonging to the mechanism can be found, most of them in more ways than one. The only difficulty connected with the operation is the choice of the order in which to take the points, as there are generally some which must precede others. It is not possible to lay down general rules for this, at least in any such form as to be practically useful. A little practice and experience, however, reduces this difficulty to very small dimensions.

We shall assume, in the following sections, that the virtual centre of any link in a mechanism relatively to any other can always be found, and in ordinary cases, where the methods of finding the point are those considered in this section, we shall merely give the necessary construction without special proof. We shall only give the proof in cases of some special difficulty, or where the use of higher forms of elements renders the construction in appearance—although not in reality—somewhat different from that generally adopted.

§ 13. DIRECTIONS OF MOTION IN MECHANISMS.

To find the direction in which any point of a mechanism is moving at any instant is now a very simple matter. Every point in each link is moving, relatively to any other link, at right angles to the line joining it to the virtual centre for the relative motion of the two links concerned. This line, the virtual radius of the point, can be drawn in every case, as we have seen, and we obtain at once the direction in which the point is moving by drawing a line at right angles to it. The construction is so simple that it requires no further explanation. It is illustrated in Fig. 34, where B_1, B_2 and B_3 are points of the link b, which is shown of general form in order to take points not lying on its axis. The lines b_1, b_2 and b_3 show the direction in which these points are moving relatively to the link d.

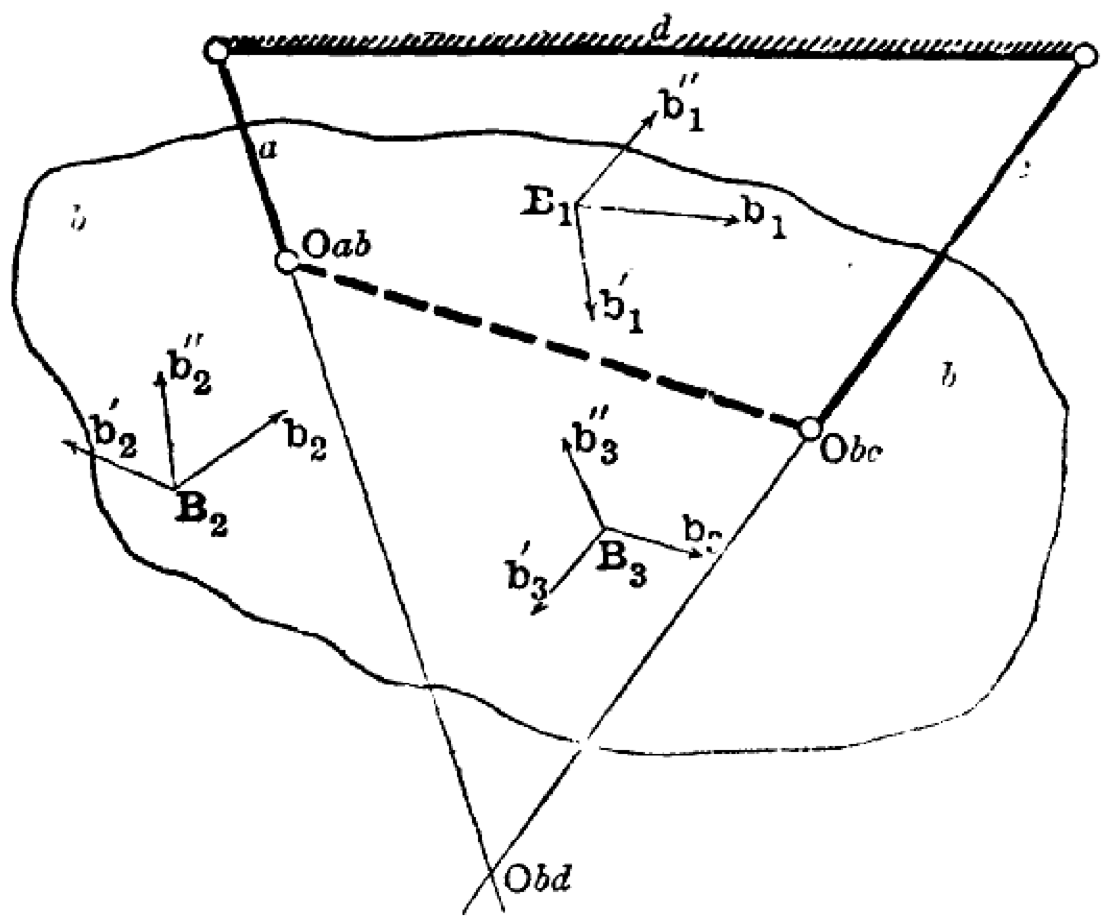

FIG. 34.

Their directions of motion relatively to a and to c are indicated by the lines b'_1, b'_2, b'_3, and b''_1, b''_2, and b''_3 respectively.