CHAPTER II.

PLANE MOTION.

§ 4. RELATIVE POSITION IN A PLANE.

We have defined motion, so far as we are now studying it, as change of position. We have seen also that we have to consider only the change in the position of one body relatively to another, and not the absolute motions of bodies. We shall now commence the more detailed treatment of this branch of our subject.

It is necessary first to examine the general conditions by which *relative position* is or may be determined. Just as the absolute *motion* of a body in space is a matter which does not concern us, so the absolute *position* of a body in space or of a figure in a plane is indifferent to us. We can assume a point or a figure stationary in any part of the plane, our object is solely to examine the position of others relatively to it.

Starting then with the notion of a fixed point in a plane, we have first the proposition that **the position of one point relatively to another is determined solely by the distance between them.** It is entirely unaffected by the *position* of the line joining them. Thus in Fig. 8, the points A and A_1, which are at the same distance from P, have the same position relatively to it, and, generally, all points in a circle occupy the same position relatively to its centre for the same reason. A point having no angular magnitude, that is, no *sides*, there cannot be any differences of angular position relatively to it. It is evident, however, that the points AA_1

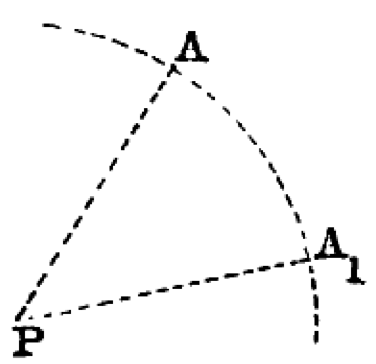

FIG. 8.

&c., occupy different positions in or relatively to the plane in which they are. We see therefore that **the position of a point in a plane is not determined by its position relatively to a point in that plane.**

A *line* is fully determined if two of its points be known. **The position of a line relatively to a point is therefore known if the positions of two of its points relatively to the fixed point be known.** These

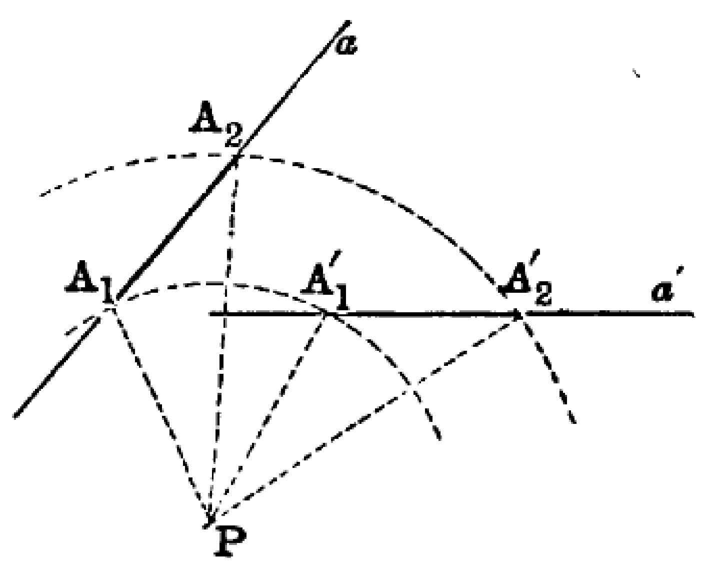

FIG. 9.

positions are determined, as mentioned in the last paragraph, solely by distances from the fixed point. As long as these distances are the same the position of the line relatively to the point is the same also. Thus in Fig. 9, where $A_1P = A'_1P$

and $A_2P=A'_2P$, the position of the line A_1A_2 relatively to the point P is the same as that of $A'_1A'_2$ relatively to the same point. But these lines are in different positions in the plane—hence **the position of a line relatively to a plane is not determined by its position relatively to a point in the plane.**[1]

The position of a *point* relatively to a *line* may be determined in two ways. It is known (i) **if its distances from two points of the line be known,** (ii) **if the positions of the lines joining it to two points of the line be known.** Thus in Fig. 10 the position of the point A

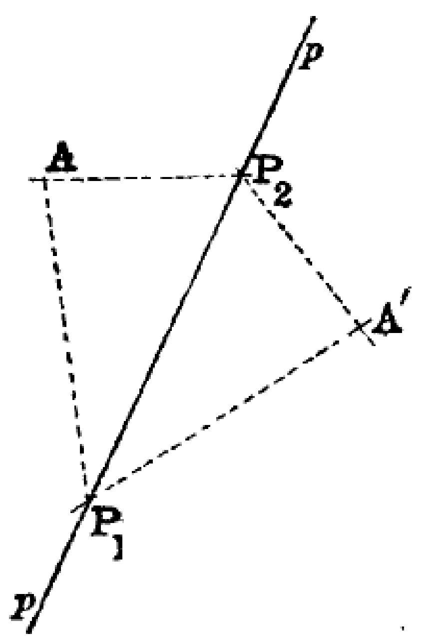

FIG. 10.

relatively to the line P_1P_2 is determined by (i), if the distances AP_1 and AP_2 be known, or by (ii) if the angles AP_1P_2 and AP_2P_1, made by AP_1 and AP_2 at the points P and P_2 of the line, be known. But we can always find two points in the plane, one on each side of the line, which shall satisfy any given conditions either in (i) or (ii). The point A', for instance occupies the same position relatively to P_1P_2 as A. A point may therefore occupy two

[1] It may be noticed in passing that the theorems just given are equally true whether or not all the points or lines are in the same plane. They hold good, that is, for spheric equally with plane motions.

positions in the plane for all positions which it can take relatively to any line in the plane, so that its position in the plane is not absolutely determined by its position relatively to a line in the plane.

We can, however, adopt some simple convention to distinguish between the two parts into which the line divides the plane; taking distances measured from P_1P_2 as positive to the one side and negative to the other, for instance. If we suppose this to be done, the symmetrical positions A and A' can be distinguished from each other, and the position of A in the plane is by this means determined when its position relatively to the line P_1P_2 is known.

The position of one line relatively to another in the same plane is known if the positions of two points in the first are known relatively to two points in the second. Here again we have an indeterminateness of the same kind as in the last case. A line may occupy two different positions in the plane, as A_1A_2 or $A'_1A'_2$ Fig. 11, and yet be in the same

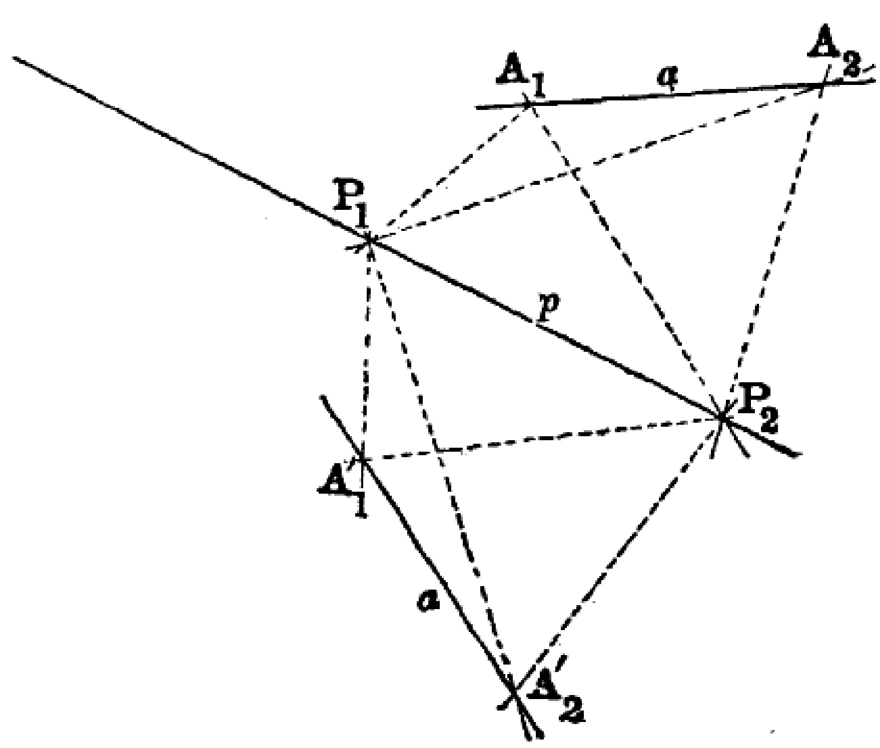

FIG. 11.

position relatively to a line P_1P_2 in the plane. If these positions be distinguished by such a convention as that just alluded to, however, the indeterminateness disappears, and we may say that **the position of a line in a plane is determined by its position relatively to any other line in the plane.**

If *a*, Fig. 12, be any given plane figure, and *A B* any two points in that figure, then if we know the positions of

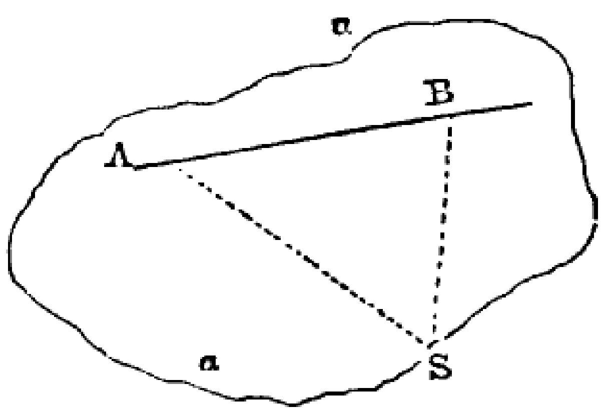

FIG. 12.

these two points we know the positions of all the others. For any other point, as *S*, can be found at once as the vertex of a triangle of which the magnitudes of all three sides (as *S A*, *A B*, *B S*) are known. **The position of a plane figure in a plane is therefore known if the positions of two points—that is, of a line—in it be known relatively to two points in the plane.**

If we discard the convention of positive and negative alluded to above, the position of a point in, *i.e.* relatively to, a plane is known only if its position relative to *three* other points in the plane, not in the same straight line, be known. Similarly the position of a line, and consequently of a plane figure, in a plane, is only completely determined if the positions of two of its points relatively to three points in the plane—not in the same straight line, be known. For our purposes, however, the two points will generally be sufficient, it is seldom that the circumstances of the case leave any doubt as to which of the two possible positions is the required one.

§ 5. RELATIVE MOTION IN A PLANE.

We have seen in the last section the conditions necessary to determine the relative positions of points, lines and figures in a plane. The *motion* of a point or line, however, is represented to us by the series of different *positions* which it occupies relatively to another point or line, &c. Each one of these is determined by the same conditions, so that the conditions which determine the *position* of the point or line relatively to any other, determine also its *motion* relatively to that other. We get therefore,—in most cases by little more than verbal alteration,—the following propositions as to relative motion in a plane, corresponding to those of the last section as to relative position.

One point can move relatively to another only along the line joining them. Thus in Fig. 13, A

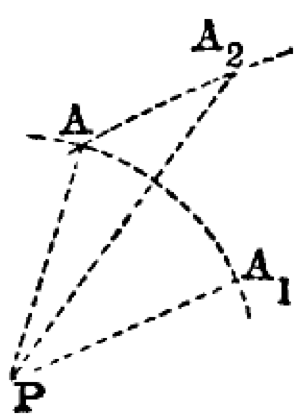

FIG. 13.

does not move relatively to P in moving to A_1, because every point in AA_1, its path of motion, has the same position relatively to P. In moving from A to A_2, however, A moves through the distance $PA_2—PA$ relatively to P.

The motion of a line relatively to a point is determined by the motion of two points in it relatively to that point. Each of these points can move, relatively to the fixed point, only along the line joining them. We see then that (just as in the case of position) the motion of a point or a line relatively to a plane is not determined by its motion relatively to a point in that plane. If a line turn about a point, for example, it remains stationary relatively to that point, although it is in continuous motion relatively to the plane.

The motion of a point relatively to a line is determined by its motion relatively to two points of the line.

The motion of a line relatively to a plane in which it moves (or to a line in that plane), is determined by the motions of two points in the one relatively to two points in the other.

And lastly **the motion of any plane figure relatively to its plane is determined by the motions of any two points,** *i.e.* **of a line, in it**

The last theorem may be stated also in another way. The figure being supposed rigid, no point in it can move relatively to any other,—all points in it, therefore, must have the same motion. But this motion is that of any line in it. When we have given, then, the motions of any two points whatever in a figure, we know the motion of the figure, and we know also that the motion of every other point in the figure is the *same* (in the sense already explained) as the known motions of the two arbitrary points with which we started.

We have already seen that when a body has plane motion the whole motion of the body is known when that of any plane section of it, moving in its own plane, is known :— the motion of the section or figure represents that of the whole body. But we have now seen further that the (plane) motion of a figure is known if the motions of two of its points be known. **The plane motion of a body,**

therefore, is known if the motion of any two points, that is of a line, in any of its sections parallel to the plane of motion, be known, and all the theorems just enunciated as to the determination of the motion of a line apply equally and absolutely to the determination of the plane motion of the body to which that line belongs. Thus for instance, the motion of the whole body shown in Fig. 2, is determined by that of any such plane section of it as the one shaded in the figure, and the motion of that section again is determined by the motion of any two points in it.

§ 6. DIRECTION OF MOTION.

We have been considering motion as a sequence of changes of position, each of finite extent. Each such change occupies some finite interval of time, at the beginning and end of which the body occupies different positions. Instead, however, of considering completed changes of position in this way, it is often necessary for us to examine the change of position which a body is actually undergoing *at some particular instant.* This is called the **instantaneous motion** of the body.

As the body moves every point in it describes some curve in the plane, and it is sometimes convenient to use the name *point-paths* for such curves. To know the whole motion of the body we must know these point-paths, or as many of them as give us the means of knowing all the rest; to know its instantaneous motion we require only to know the *direction* of the point-paths at the given instant. By the direction of the point-path at any instant is meant the direction in which the point which describes that path is moving at that instant, that is, the direction of a line joining the point with the next consecutive point of the curve it is describing, which is, of course, infinitely near to the first. But a line which joins two consecutive points of a curve is called a tangent to the curve. Two such points cannot be any finite distance apart, or it would be possible to find another point between them, and they would not be consecutive. We therefore assume the distance between them to be infinitely small, or in other words we assume them to coincide. A tangent therefore,—a line joining two consecutive points of the curve,—is by definition a line passing through two points of the curve, but it differs from all other lines which have the same property in that these points are coincident. Fig. 14 may make this clearer. Suppose the

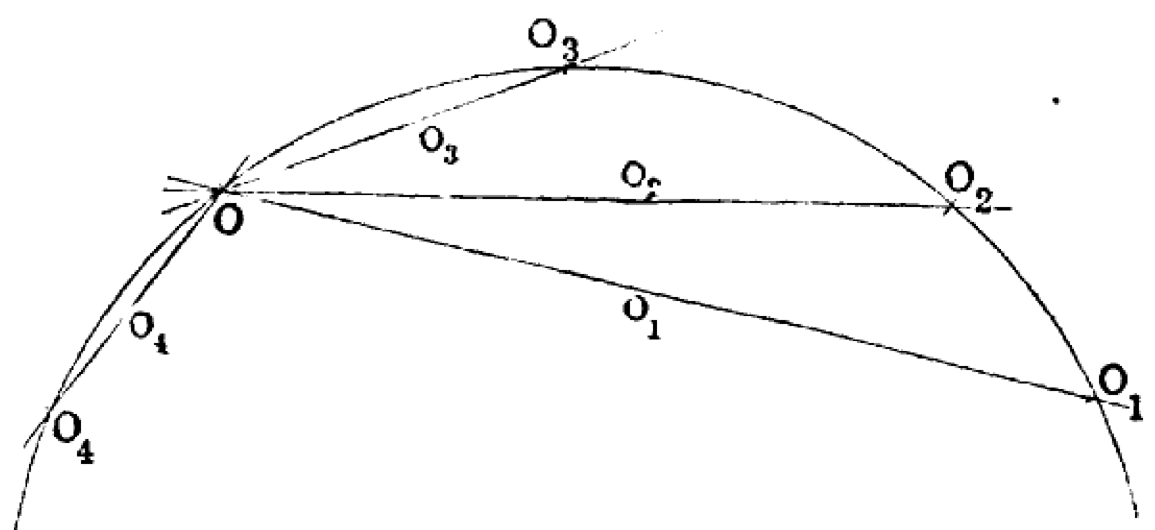

FIG. 14.

line o to turn from o_1 to o_2, o_3, etc., about the point O in the curve. It cuts the curve always in O and in some other point, and this point moves continuously along the curve, taking the positions, $O_1 O_2 O_3 \ldots \ldots O_4$ &c. In doing so the second point must have passed through O itself, for it has passed over from one side of O to the other. When the line occupies this central position, its two points of intersection with the curve are said to coincide at O, and it is called the tangent to the curve at O.

When a point then is moving in any curve, its direction of motion at any instant coincides with the direction of the tangent to the curve drawn through the point.

By reasoning similar to that adopted in the last two sections we have then at once the following propositions relating to instantaneous motion. **The instantaneous motion of a point is known if its direction of motion,** *i.e.*, **the tangent to its path, be known for the given instant.** Here again we have conditions similar to those which were examined in §§ 4 and 5; the path of the point relatively to another point is not, in general, the same as its path relatively to the plane. Its instantaneous motion will differ, therefore, according to the standard relatively to which it is observed, just as its change of position does.

The instantaneous motion of a line is known if the directions of motion of (or tangents to the paths of) any two of its points be known for the given instant. In both cases it is only point-paths or directions relatively to the plane with which we need concern ourselves at present.

We have seen that the motion of any plane figure in the plane can be fully determined from the motion of any two of its points. This is as true in the case of instantaneous motion as in the case of finite change of position. The former differs from the latter only in that the changes of position are regarded in it as being indefinitely small. It requires two points (assumed to be indefinitely near together) to determine each tangent, and these are simply the two consecutive positions of one point when its change of position has become infinitely small. We get, therefore, the important proposition that **the instantaneous motion of**

a plane figure in its plane is determined by that of any two of its points. And from this follows the very important corollary that **the directions of motion of all points in a figure are fixed when those of two points in it are fixed.**

The direction of motion of a point, in the sense in which we have been using the word, is given us by a line. But a point may be said to move in either of two "directions" along this line. To avoid any indistinctness from this double use of one word we shall restrict "direction" to the former meaning, using "sense" for the latter. A line, then, determines a *direction*, while along that line a point may move in either of two *senses*, which we shall often have to distinguish. In writing of them we may call one positive, and the other negative; in figures the particular sense of motion can be shown always by an arrow. The word "sense" is used similarly in reference to a line turning about a point. It may be turning either clock-hand-wise, or in the opposite *sense.*

§ 7. THE INSTANTANEOUS OR VIRTUAL CENTRE.

The instantaneous motion of any point is completely known, as we have now seen, when the *direction* of its motion is known,—it is therefore quite independent of the form of the curve in which the point is moving. A point, therefore, of which the position and the direction of motion are known may have for its actual path any one of the infinite number of different curves which can be drawn touching the given direction line in the given point. For example, in Fig. 15 a point O is moving at a particular instant in the direction of the line p. But its actual point-path may be o_1, o_2, o_3, o_4, or any curve whatever which is touched by the line p at the given point. This fact enables us to simplify our problems enormously, so far as they have

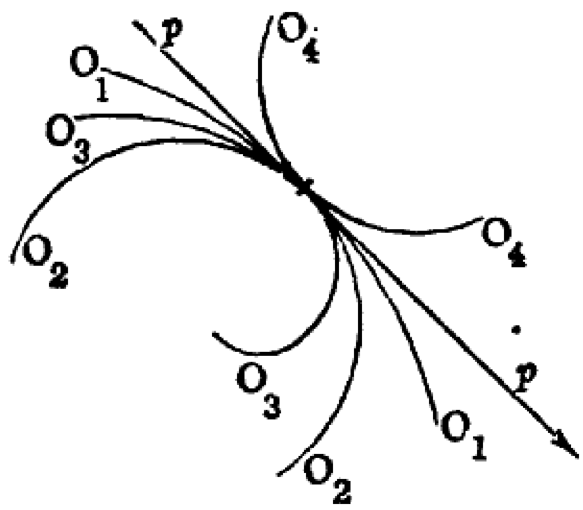

FIG. 15.

to do with instantaneous motion. In all machines certain leading and very important points have very simple motions,—rotation or translation,—but in all machines except the very simplest there occur constrained motions by no means so simple as these, and not unfrequently really complex. Sometimes we are concerned directly with the form of the motion in such cases, and then it has simply to be worked out in the most direct way possible. More frequently, however, as will be found, we are concerned only with the instantaneous motion of each body forming the machine, and with the directions in which its points are moving at some particular instant. In this case it becomes very easy to substitute for the actual complex motion an imaginary simple one which for the instant is identical with it, and which admits of treatment of the most direct possible kind. This we can do in the following way :—Let a be any plane figure (Fig. 16), and A and B any two points in it, of which the directions of motion, a and b respectively, relatively to the plane β, are known. By these data the instantaneous motion of the whole body is, as we have seen, determined. Let AO and BO be perpendiculars drawn to a and b at the points A and B. Then about every point in AO we can draw a circle touching a in A, and the instantaneous motion of A, whatever its

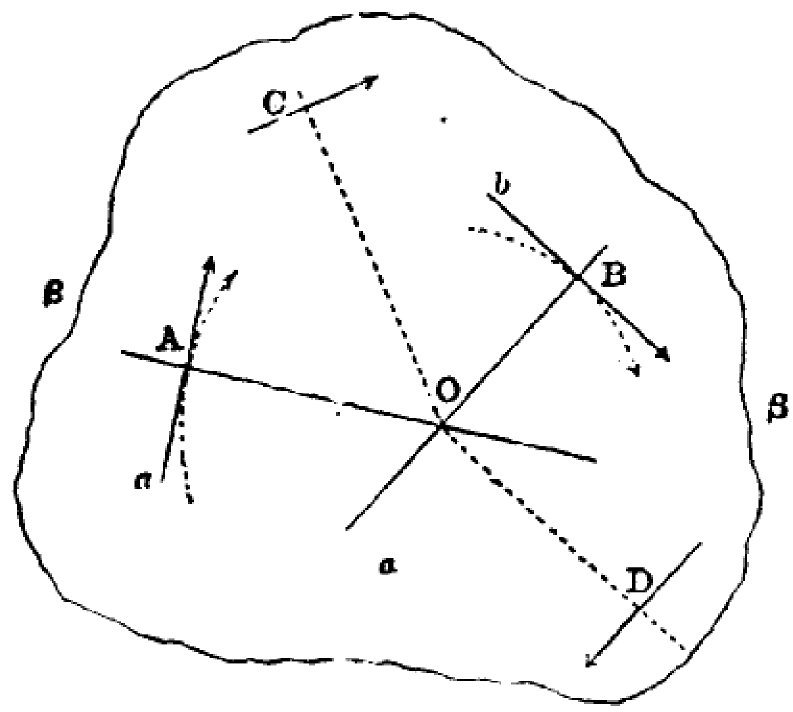

FIG. 16.

actual path, is the same as if it were moving in any one of these circles. Similarly we can draw a circle about every point in BO touching b in B, and any one of these circles if it were the path of B would give it the same instantaneous motion as that which it actually has. But AO and BO, being lines in the same plane, must have one point in common, their intersection or join,—here the point O. If, therefore, A and B were both moving round this point as a centre their instantaneous motion would remain just what it is,—for the lines a and b would still be the direction of motion of A and B, or tangents to their paths; but these paths would now be circles having O as their common centre. It has already been shown that the plane motion of any figure is the same as that of any line in it. In this case the line AB is for the instant simply rotating about the point O,—**the motion of the whole figure α, therefore, is simply a rotation about the point O, which**