11 ALGEBRAIC METHODS OF SYNTHESIS USING COMPLEX NUMBERS

11-1 VELOCITY AND ACCELERATION SYNTHESIS BY COMPLEX NUMBERS[1]

The problems of synthesis considered in the previous chapter have been solved by using the displacement equation relating the input and output variables in terms of design parameters. Values of the design parameters for which this input-output relation satisfied given conditions of motion were found analytically without further reference to the geometry of the problem. The method to be considered here follows the same general pattern, but the displacement equation to be used will be written in terms of complex numbers. The use of complex numbers makes it possible to consider not only angles and distances, as rotations of cranks or translations of sliders, but also vectors, to express analytically the arbitrary motions of points in a plane.

In the four-bar linkage O_AABO_B (Fig. 11-1) the frame (link 4) is stationary, but the other three links (1, 2, and 3) possess angular velocities ω_1, ω_2, and ω_3 and angular accelerations α_1,

[1] Much of the material of this section appeared in *Machine Design*, Mar. 20, 1958, and is reprinted by courtesy of the Penton Publishing Company, Cleveland.

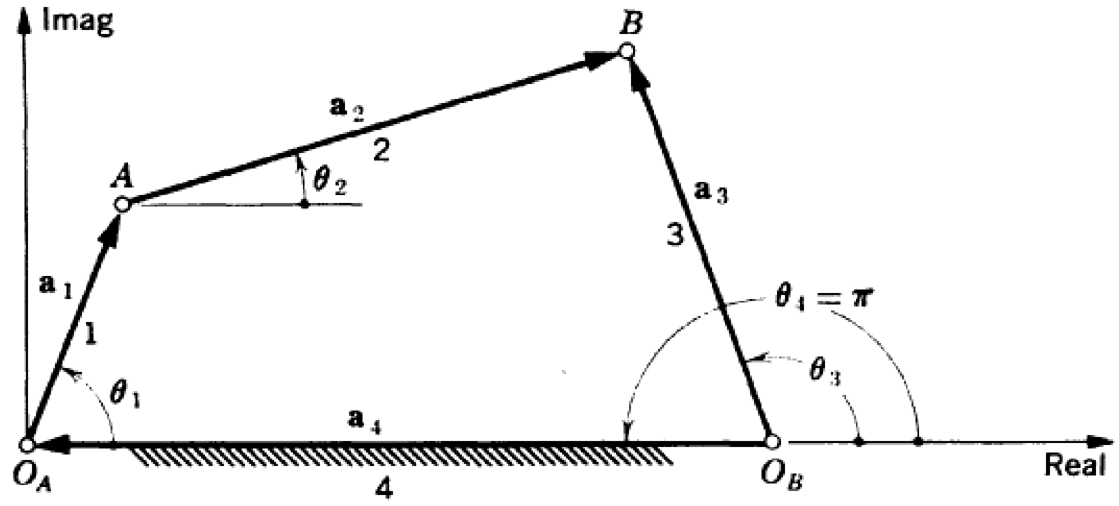


FIGURE 11-1 Four-bar linkage and vector polygon.

α_2, and α_3. The problem considered in this section is to find not only the link lengths a_1, a_2, a_3, and a_4 but also the relative positions of the links satisfying angular velocity and acceleration specifications.

The four-bar linkage O_AABO_B may be considered to be defined by four vectors, since four points are involved. In fact, the linkage will now be taken to consist of a closed vector polygon (Fig. 11-1), for which we may write

$$\mathbf{a}_4 + \mathbf{a}_1 + \mathbf{a}_2 = \mathbf{a}_3$$

or

$$\mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3 + \mathbf{a}_4 = 0 \tag{11-1}$$

Here the relation of one revolute connection to another is given by the directed distances $\mathbf{a}$ (vectors).

Since a vector such as $\mathbf{a}$ may be written $\mathbf{a} = ae^{i\theta}$, in which a is a distance and θ a counterclockwise angle measured from the real axis, the vector equation of the polygon may be written in complex-number form,

$$a_1e^{i\theta_1} + a_2e^{i\theta_2} - a_3e^{i\theta_3} + a_4e^{i\pi} = 0$$

The last term may be simplified, for $e^{i\pi} = -1$, whence $\mathbf{a}_4 = -a_4$, and

$$a_1e^{i\theta_1} + a_2e^{i\theta_2} - a_3e^{i\theta_3} - a_4 = 0 \tag{11-2}$$

This equation represents the space relation of the points O_A, A, B, and O_B, the points of connection between links.

On differentiating with respect to time, setting $d\theta/dt = \omega$, and ordering terms,

$$i(a_1\omega_1)e^{i\theta_1} + i(a_2\omega_2)e^{i\theta_2} - i(a_3\omega_3)e^{i\theta_3} - a_4(0) = 0 \tag{11-3}$$

The terms are recognized as defining the linear velocities of the points. This is then the velocity relation and represents the velocity-vector diagram.

A second differentiation yields, after setting $d\theta/dt = \omega$, $d\omega/dt = \alpha$ and ordering,

$$i(a_1\alpha_1)e^{i\theta_1} + i^2(a_1\omega_1^2)e^{i\theta_1} + i(a_2\alpha_2)e^{i\theta_2} + i^2(a_2\omega_2^2)e^{i\theta_2} - i(a_3\alpha_3)e^{i\theta_3} - i^2(a_3\omega_3^2)e^{i\theta_3} - (a_4)0 = 0 \quad (11\text{-}4)$$

Here the factors $i(a\alpha)$ and $i^2(a\omega^2)$ are recognized as associated with the linear acceleration components of the points. Equation (11-4) is thus an acceleration relation involving the points; it represents the acceleration-vector diagram.

A more convenient form for future manipulation results on dividing by i and rearranging,

$$(\alpha_1 + i\omega_1^2)a_1e^{i\theta_1} + (\alpha_2 + i\omega_2^2)a_2e^{i\theta_2} - (\alpha_3 + i\omega_3^2)a_3e^{i\theta_3} - (a_4)0 = 0 \quad (11\text{-}5)$$

Assembling Eqs. (11-2), (11-3), and (11-5) as a group and replacing each $ae^{i\theta}$ by its vector $\mathbf{a}$ yields

$$\begin{aligned} 1\mathbf{a}_1 + 1\mathbf{a}_2 - 1\mathbf{a}_3 + 1\mathbf{a}_4 &= 0 \\ \omega_1\mathbf{a}_1 + \omega_2\mathbf{a}_2 - \omega_3\mathbf{a}_3 + 0\mathbf{a}_4 &= 0 \\ (\alpha_1 + i\omega_1^2)\mathbf{a}_1 + (\alpha_2 + i\omega_2^2)\mathbf{a}_2 - (\alpha_3 + i\omega_3^2)\mathbf{a}_3 + 0\mathbf{a}_4 &= 0 \end{aligned} \quad (11\text{-}6)$$

This is a system of three homogeneous equations in four unknowns, the vectors (or complex numbers) $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$, in which some of the coefficients are complex numbers which involve the ω and α values of the links. Since this system, consisting of only three equations, involves four unknowns, one of the unknowns, $\mathbf{a}_4$ for example, may be chosen arbitrarily and the system rewritten as

$$\begin{aligned} 1\mathbf{a}_1 + 1\mathbf{a}_2 - 1\mathbf{a}_3 &= -\mathbf{a}_4 \\ \omega_1\mathbf{a}_1 + \omega_2\mathbf{a}_2 - \omega_3\mathbf{a}_3 &= 0 \\ (\alpha_1 + i\omega_1^2)\mathbf{a}_1 + (\alpha_2 + i\omega_2^2)\mathbf{a}_2 - (\alpha_3 + i\omega_3^2)\mathbf{a}_3 &= 0 \end{aligned} \quad (11\text{-}7)$$

The solution may then be carried out by determinants. With

$$\mathbf{D} = \begin{vmatrix} 1 & 1 & -1 \\ \omega_1 & \omega_2 & -\omega_3 \\ \alpha_1 + i\omega_1^2 & \alpha_2 + i\omega_2^2 & -(\alpha_3 + i\omega_3^2) \end{vmatrix} \quad (11\text{-}8)$$

the determinant of the system, the unknowns $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ are expressed in complex-number form as

$$\begin{aligned} \mathbf{a}_1 &= \frac{-\mathbf{a}_4[-\omega_2(\alpha_3 + i\omega_3^2) + \omega_3(\alpha_2 + i\omega_2^2)]}{\mathbf{D}} \\ \mathbf{a}_2 &= \frac{-\mathbf{a}_4[-\omega_3(\alpha_1 + i\omega_1^2) + \omega_1(\alpha_3 + i\omega_3^2)]}{\mathbf{D}} \\ \mathbf{a}_3 &= \frac{-\mathbf{a}_4[\omega_1(\alpha_2 + i\omega_2^2) - \omega_2(\alpha_1 + i\omega_1^2)]}{\mathbf{D}} \end{aligned} \quad (11\text{-}9)$$

The difficulty with this solution lies in the complexity of the third-order determinant $\mathbf{D}$. This difficulty, however, may be overcome if the arbitrary $\mathbf{a}_4$ is taken proportional to the determinant $\mathbf{D}$ itself. With

$$\mathbf{a}_4 = -\mathbf{D} \tag{11-10}$$

the above values of $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ become independent of $\mathbf{D}$ and are expressed in simple form as shown in Table 11-1. The value of $\mathbf{a}_4$, which still remains to be found, may then be obtained from (11-1),

$$\mathbf{a}_4 = -\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3 \tag{11-11}$$

In Table 11-1, the links are defined as complex numbers of the form $\mathbf{a} = c + id$, whose real and imaginary parts are themselves defined by the angular velocities and accelerations specified for the links.

When the vectors represented by the complex numbers are assembled in order (see examples), they will define the proportions of a mechanism having the specified velocity and acceleration values. In general, the fixed link (in this case $\mathbf{a}_4$) will not "come out horizontal," even though the basic sketch (Fig. 11-1) placed it that way; $\mathbf{a}_4$ will generally have an i component, which means a position rotation. However, all links will be rotated by the same amount. This is a consequence of having set $\mathbf{a}_4 = -\mathbf{D}$ in the solution of the system of equations (11-6) instead of setting $\mathbf{a}_4$ real as shown in Fig. 11-1.

If the inputs to the definitions of Table 11-1 are incompatible with physical reality, then no mechanism will result in the sense that a vector of zero length will be given. The definitions cannot be called upon with-

Table 11-1 FOUR–BAR LINKAGE

VECTOR OR LINK a	REAL COMPONENT c	i COMPONENT d	COMPLEX FORM $a = c + id$	LENGTH $a = \sqrt{c^2 + d^2}$
a_1	$c_1 = \omega_3\alpha_2 - \omega_2\alpha_3$	$d_1 = \omega_2\omega_3(\omega_2 - \omega_3)$	$a_1 = c_1 + id_1$	$a_1 = \sqrt{c_1^2 + d_1^2}$
a_2	$c_2 = \omega_1\alpha_3 - \omega_3\alpha_1$	$d_2 = \omega_3\omega_1(\omega_3 - \omega_1)$	$a_2 = c_2 + id_2$	$a_2 = \sqrt{c_2^2 + d_2^2}$
a_3	$c_3 = \omega_1\alpha_2 - \omega_2\alpha_1$	$d_3 = \omega_1\omega_2(\omega_2 - \omega_1)$	$a_3 = c_3 + id_3$	$a_3 = \sqrt{c_3^2 + d_3^2}$
a_4	$c_4 = c_3 - c_1 - c_2$	$d_4 = d_3 - d_1 - d_2$	$a_4 = c_4 + id_4$	$a_4 = \sqrt{c_4^2 + d_4^2}$

out some regard for sensible and compatible magnitudes, but poor estimates are quickly found and revised.

In making entries into the definitions of the components of Table 11-1, it must be remembered that the positive directions of angular velocity and acceleration are the same as the positive directions for the angles from which they derive; positive is counterclockwise.

Example 1 To determine the links of a four-bar mechanism that will in one of its positions satisfy the following specifications:

$$\begin{aligned} \omega_1 &= 8 \text{ rad/sec} & \alpha_1 &= 0 \\ \omega_2 &= 1 \text{ rad/sec} & \alpha_2 &= 20 \text{ rad/sec}^2 \\ \omega_3 &= -3 \text{ rad/sec} & \alpha_3 &= 0 \end{aligned}$$

Substituting values into the definitions of Table 11-1,

$$\begin{aligned} \mathbf{a}_1 &= -3(20) - 1(0) + i(1)(-3)(1+3) \\ &= -60 - i(12) \\ a_1 &= 61.19 \text{ units} \\ \mathbf{a}_2 &= 8(0) - (-3)(0) + i(-3)(8)(-3-8) \\ &= 0 + i(264) \\ a_2 &= 264.00 \text{ units} \\ \mathbf{a}_3 &= 8(20) - 1(0) + i(8)(1)(1-8) \\ &= 160 - i(56) \\ a_3 &= 169.52 \text{ units} \\ \mathbf{a}_4 &= 160 - (-60) - 0 + i(-56 + 12 - 264) \\ &= 220 - i(308) \\ a_4 &= 378.46 \text{ units} \end{aligned}$$

The vectors represented by the complex numbers are shown in Fig. 11-2*a*. The mechanism is formed by assembling the vectors in sequence, starting with $\mathbf{a}_4$ (Fig. 11-2*b*). The proportions of a mechanism responding to the specified motion characteristics are now on a relative basis. The mechanism has appeared with a rotation, a consequence of a mathematical manipulation which is of no importance to our physical problem. The relative lengths of the bars and their terminal points have been established as functions of the specified ω and α values. Needless to say, the bar $\mathbf{a}_4$ must always be the fixed link. This example may be compared with the result obtained by a different method in Sec. 10-4.

Dead Points

A dead point occurs when the follower is momentarily at rest just prior to reversing its direction of rotation, that is, when $\omega_3 = 0$. With continuously rotating crank, the crank and coupler are either (1) extended in a straight line or (2) folded over each other into a straight line. Condition

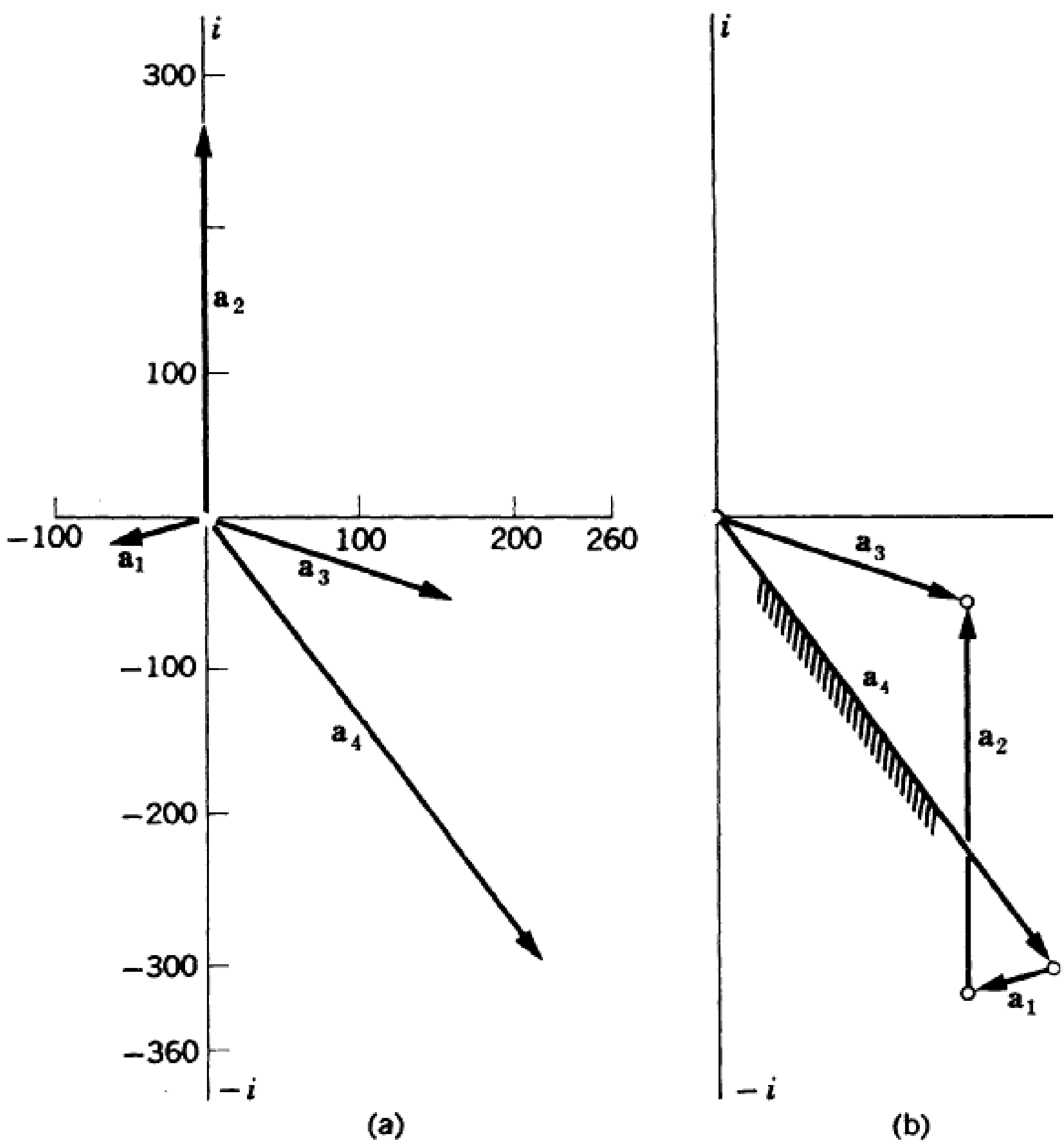


FIGURE 11-2 Example.

1 calls for $\theta_1 = \theta_2$, whence $d_1/d_2 = c_1/c_2$ or $d_1 = d_2c_1/c_2$. Condition 2 requires that $\theta_1 + 180 = \theta_2$, for which $d_1 = d_2c_1/c_2$ also applies. If

$$\omega_1 \neq 0 \qquad \omega_2 \neq 0 \qquad \omega_3 = 0 \qquad \alpha_1 \neq 0 \qquad \alpha_2 \neq 0 \qquad \alpha_3 \neq 0$$

the link values reduce to

$$\begin{aligned}
\mathbf{a}_1 &= -\omega_2\alpha_3 + 0\\
\mathbf{a}_2 &= \omega_1\alpha_3 + 0\\
\mathbf{a}_3 &= \omega_1\alpha_2 + i\omega_1\omega_2(\omega_2 - \omega_1)\\
\mathbf{a}_4 &= \omega_1\alpha_2 + \omega_2\alpha_3 - \omega_1\alpha_3 + i\omega_1\omega_2(\omega_2 - \omega_1)
\end{aligned}$$

Example 2 Condition 1, crank and coupler *extended* in a straight line. Here ω_2 will be negative, with α_2 and α_3 both positive. Consider a mechanism in which $\omega_1 = 3$, $\omega_2 = -2$, $\omega_3 = 0$, $\alpha_1 = 0$, $\alpha_2 = \frac{8}{3}$, and $\alpha_3 = 8$ (radian-second units). The links are then $\mathbf{a}_1 = 16$, $\mathbf{a}_2 = 24$, $\mathbf{a}_3 = 8 + i30$, and $\mathbf{a}_4 = -32 + i30$ units. This mechanism is shown in Fig. 11-3.

Example 3 Condition 2, crank and coupler *folded* over each other into a straight line. Here ω_2 and α_2 will be positive, with α_3 negative.

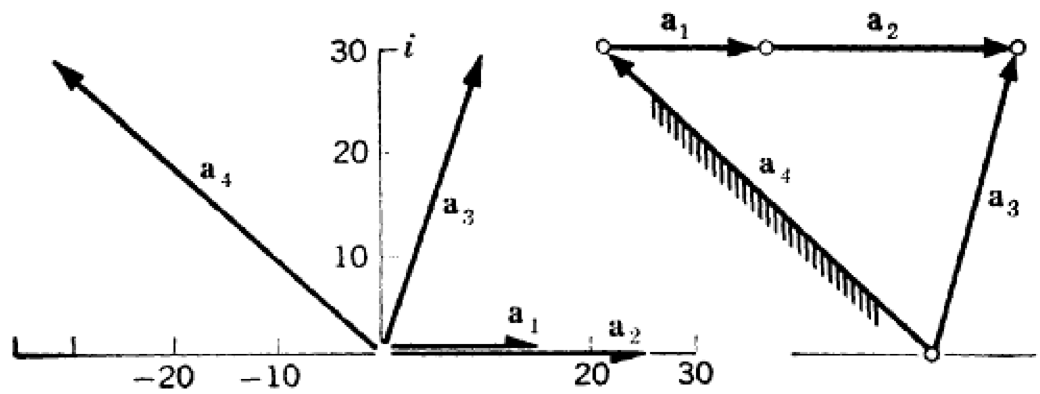


FIGURE 11-3 Dead point: crank and coupler extended in straight line.

Consider a mechanism in which $\omega_1 = 3$, $\omega_2 = 1.5$, $\omega_3 = 0$, $\alpha_1 = 0$, $\alpha_2 = 1.7$, and $\alpha_3 = -3.7$ (radian-second units). The links are then $\mathbf{a}_1 = 5.55$, $\mathbf{a}_2 = -11.10$, $\mathbf{a}_3 = 5.10 - i6.75$, and $\mathbf{a}_4 = 10.65 - i6.75$ units. The mechanism is shown in Fig. 11-4.

11-2 COUPLER-CURVE SYNTHESIS: FIVE ACCURACY POINTS

The problem to be considered in this section is the synthesis of a four-bar linkage (Fig. 11-5*a*) that is to generate a coupler curve prescribed by means of the coordinates of specified accuracy points (Fig. 11-5*b*). As the coupler point passes through these accuracy points, the crank must rotate through prescribed angles $\phi_2, \phi_3, \ldots$ measured from position 1, which corresponds to the first accuracy point (Fig. 11-5*c*). The design parameters to be used in this synthesis are the link lengths a_1, a_2, a_3, a_4; the coordinates x and y of the point O_A with respect to a coordinate system Oxy; the angle θ between the line O_AO_B and the axis Ox; the distance b and the angle σ defining the coupler point to be used; and finally the initial crank angle ϕ_1. A total of 10 design parameters is thus at hand. Since each accuracy point is given by two coordinates, a maximum of five accuracy points may be specified on matching 10 coordinates with 10 design parameters.

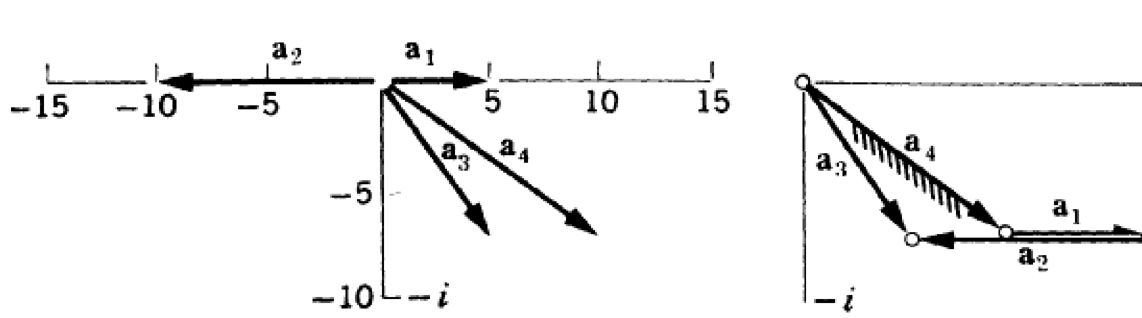
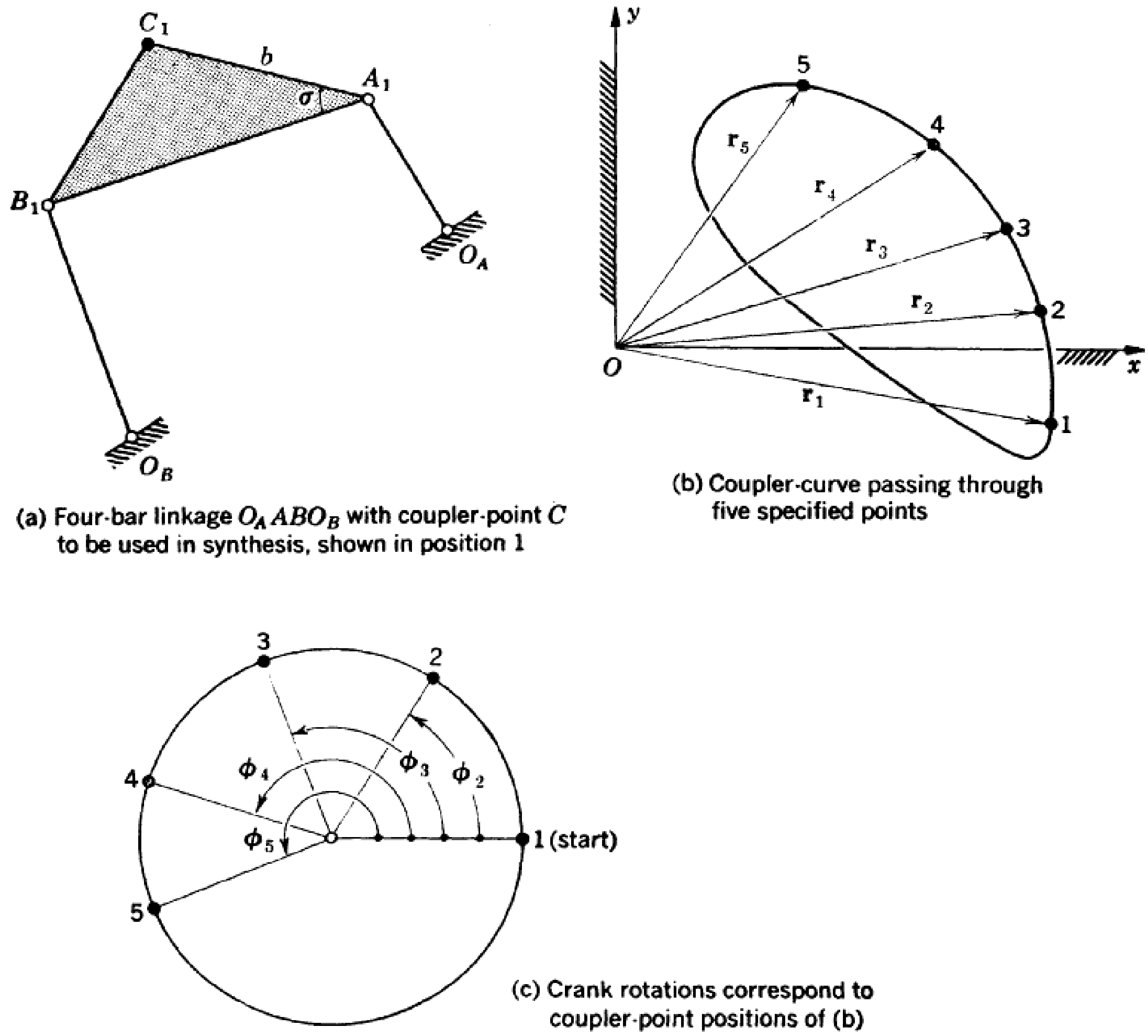


FIGURE 11-4 Dead point: crank and coupler folded over each other.



(a) Four-bar linkage $O_A ABO_B$ with coupler-point C to be used in synthesis, shown in position 1

(b) Coupler-curve passing through five specified points

(c) Crank rotations correspond to coupler-point positions of (b)

FIGURE 11-5 Notation for coupler-curve synthesis.

A solution of this five-accuracy-point coupler-curve problem has been developed by Freudenstein and Sandor. This solution, making use of complex numbers, contains such lengthy numerical calculations that it is feasible only when accomplished with a digital computer. Available on an IBM-650 program, the solution may be applied to a problem without a complete understanding of the convolutions of the method. However, the utility of any method does gain from an understanding, for, among other things, it may lead to extensions and different applications. This chapter will provide understanding by treating the procedures and speaking of the details of the solution of a rather complex problem of kinematic synthesis. As a matter of convenience, one part of the solution—the reduction of the first pair of compatibility equations requiring complicated algebraic manipulation—is treated separately (Sec. 11-3) and may be omitted in a first reading.

The radius vectors defining the accuracy points are denoted by the complex numbers

$$\mathbf{r}_j = r_j e^{i\theta_j} = r_{jx} + i r_{jy} \qquad j = 1, 2, \ldots, 5 \tag{11-12}$$

referred to an arbitrary coordinate system Oxy in the plane of the fixed link. The configuration of the linkage in position 1, its location relative to the coordinate system Oxy, and the location of the coupler point are defined by the complex numbers

$$\mathbf{z}_k = z_k e^{i\beta_k} = z_{kx} + i z_{ky} \qquad k = 1, \ldots, 7 \tag{11-13}$$

as shown in Fig. 11-6.

A rotation of the crank from position 1 to position j, already defined as ϕ_j, may be expressed by the complex number

$$\lambda_j = e^{i\phi_j}$$

Thus, the product $\lambda_j \mathbf{z}_1$ denotes the crank in position j. Similarly, rotations γ_j and ψ_j of the coupler and follower from position 1 to position j may be expressed by the complex numbers

$$\nu_j = e^{i\gamma_j} \qquad \text{and} \qquad \mu_j = e^{i\psi_j}$$

The radius vector of the coupler point in position 1 (Fig. 11-6) may now be expressed as a sum of vectors in two different ways as

$$\mathbf{r}_1 = \mathbf{z}_7 + \mathbf{z}_5 + \mathbf{z}_1 + \mathbf{z}_2 = \mathbf{z}_7 + \mathbf{z}_4 + \mathbf{z}_3$$

In position j (Fig. 11-7) the radius vector may similarly be expressed as

$$\mathbf{r}_j = \mathbf{z}_7 + \mathbf{z}_5 + \lambda_j \mathbf{z}_1 + \nu_j \mathbf{z}_2 = \mathbf{z}_7 + \mu_j \mathbf{z}_4 + \nu_j \mathbf{z}_3$$

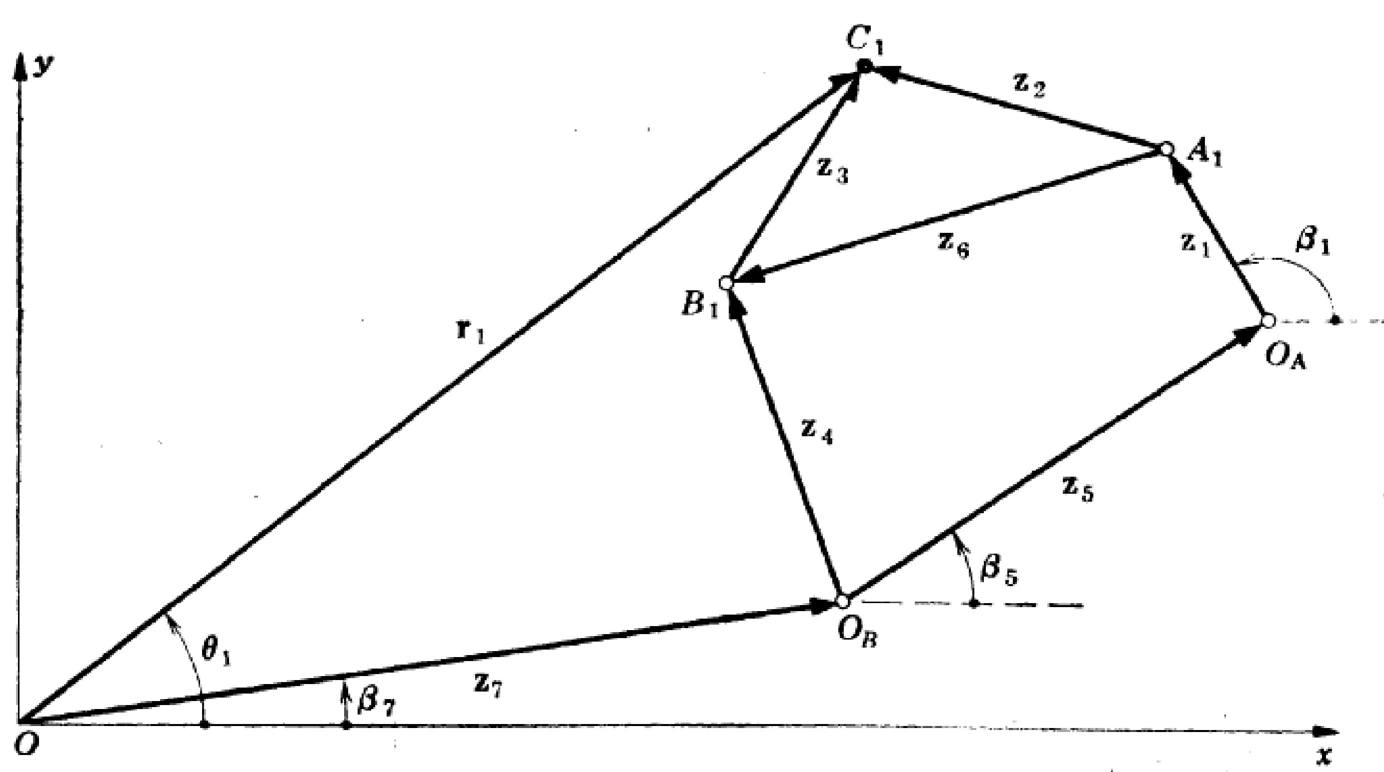


FIGURE 11-6 Vectors to be used in coupler-curve synthesis.

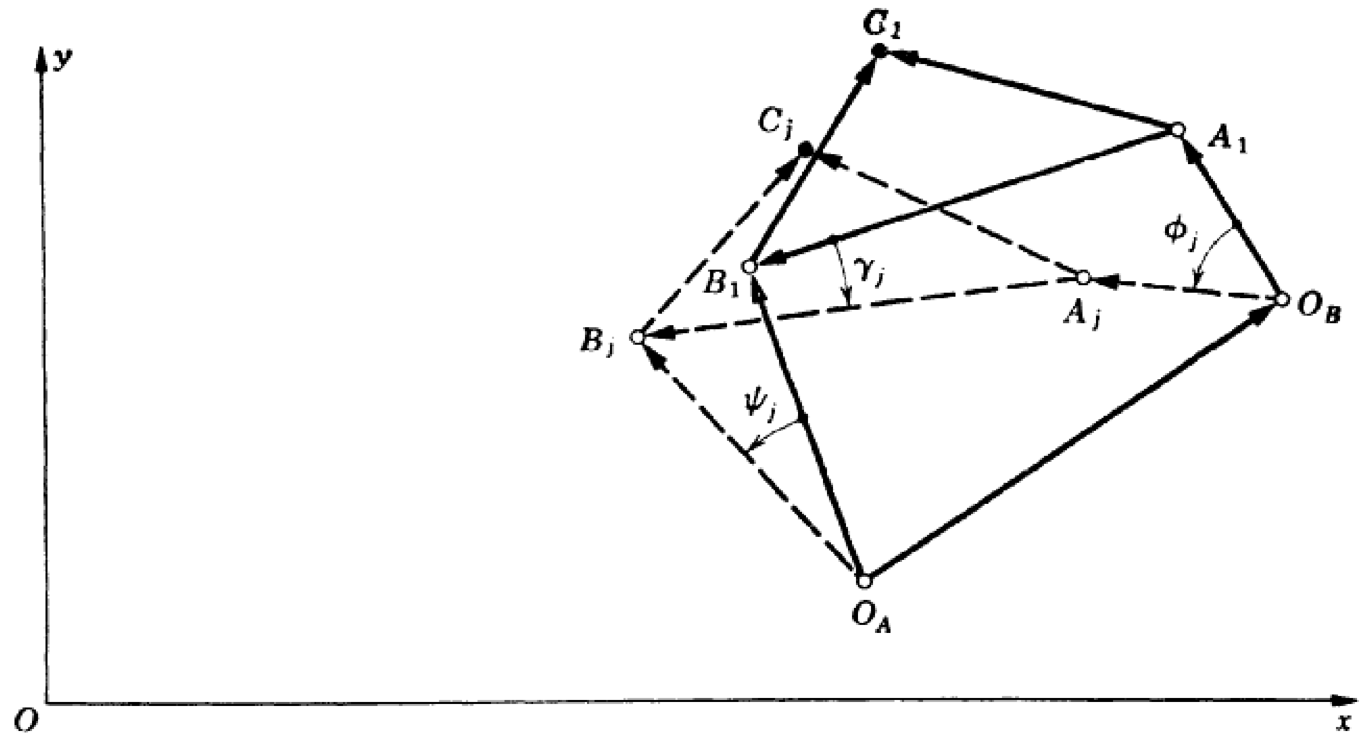


FIGURE 11-7 Displacement of linkage from position 1 to position j.

If $\boldsymbol{\delta}_j$ denotes the displacement of the coupler point from position 1 to position j, the vectors (or complex numbers) $\boldsymbol{\delta}_j$, with $j = 2$, 3, 4, and 5, are known and

$$\boldsymbol{\delta}_j = \mathbf{r}_j - \mathbf{r}_1$$

or
$$\boldsymbol{\delta}_j = \mathbf{z}_1(\lambda_j - 1) + \mathbf{z}_2(\nu_j - 1) = \mathbf{z}_4(\mu_j - 1) + \mathbf{z}_3(\nu_j - 1)$$

With five accuracy points, these last equations must hold for $j = 2, 3, 4, 5$. Since the vectors $\boldsymbol{\delta}_j$ and the coefficients λ_j are known, this condition yields two systems of equations. From this point on, the vector designation of the r, z, and δ vectors will be dropped, since it is understood that such vectors are expressed in terms of complex numbers. The two systems of equations are then

$$\begin{array}{c}(\nu_2 - 1)z_2 + (\lambda_2 - 1)z_1 = \delta_2 \\ \cdots\cdots\cdots\cdots\cdots \\ (\nu_5 - 1)z_2 + (\lambda_5 - 1)z_1 = \delta_5\end{array} \tag{11-14}$$

and

$$\begin{array}{c}(\mu_2 - 1)z_4 + (\nu_2 - 1)z_3 = \delta_2 \\ \cdots\cdots\cdots\cdots\cdots \\ (\mu_5 - 1)z_4 + (\nu_5 - 1)z_3 = \delta_5\end{array} \tag{11-15}$$

Each system involves two unknowns, z_1, z_2 and z_3, z_4, present in four equations. Since each system has only two unknowns, solutions will be possible only if the matrix of each system is of rank 2 and their third-order characteristic determinants are zero.[1] On the assumption that

$$D_1 = \begin{vmatrix} \nu_2 - 1 & \lambda_2 - 1 \\ \nu_3 - 1 & \lambda_3 - 1 \end{vmatrix} \neq 0 \qquad \text{and} \qquad D_2 = \begin{vmatrix} \mu_2 - 1 & \nu_2 - 1 \\ \mu_3 - 1 & \nu_3 - 1 \end{vmatrix} \neq 0$$

[1] See Sec. A-4 for definition of characteristic determinant.

these determinants may be taken as principal determinants for each system. Two characteristic third-order determinants may then be formed for each system as shown in Sec. A-4. The first system (11-14) will be compatible if

$$\begin{vmatrix} \nu_2 - 1 & \lambda_2 - 1 & \delta_2 \\ \nu_3 - 1 & \lambda_3 - 1 & \delta_3 \\ \nu_4 - 1 & \lambda_4 - 1 & \delta_4 \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} \nu_2 - 1 & \lambda_2 - 1 & \delta_2 \\ \nu_3 - 1 & \lambda_3 - 1 & \delta_3 \\ \nu_5 - 1 & \lambda_5 - 1 & \delta_5 \end{vmatrix} = 0 \qquad (11\text{-}16)$$

For the second system (11-15) the conditions of compatibility are

$$\begin{vmatrix} \mu_2 - 1 & \nu_2 - 1 & \delta_2 \\ \mu_3 - 1 & \nu_3 - 1 & \delta_3 \\ \mu_4 - 1 & \nu_4 - 1 & \delta_4 \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} \mu_2 - 1 & \nu_2 - 1 & \delta_2 \\ \mu_3 - 1 & \nu_3 - 1 & \delta_3 \\ \mu_5 - 1 & \nu_5 - 1 & \delta_5 \end{vmatrix} = 0 \qquad (11\text{-}17)$$

The first set, Eqs. (11-16), which guarantee the compatibility of the system of Eqs. (11-14), are called the *first pair of compatibility equations.* The solution of these equations yields values of $\nu_2, \ldots, \nu_5$ for which the system of Eqs. (11-14) may be solved. The last set, Eqs. (11-17), which guarantee the compatibility of the system of Eqs. (11-15), constitute the *second pair of compatibility equations.*

Since the known quantities are the λ's and δ's, the first pair of compatibility equations (11-16) must be solved first. Note that these equations involve complex numbers: when their real and imaginary parts are equated to zero, they yield four equations which may be solved for the four unknowns $\nu_2, \ldots, \nu_5$, which are complex numbers of unit magnitude. With these values of the ν's, the system of Eqs. (11-14) is compatible. After a solution of the first pair of compatibility equations has been obtained, the corresponding values of $\nu_2, \ldots, \nu_5$ may be substituted in the second pair (11-17), which then yield values of $\mu_2, \ldots, \mu_5$ for which the system of Eqs. (11-15) may be solved. The desired linkage is then obtained by solving the systems of Eqs. (11-14) and (11-15).

In order to solve the first pair of compatibility equations, the variables ν_5, ν_4, and ν_3 must be successively eliminated to leave ν_2 as the only unknown. The resulting equation is then reduced to algebraic form in terms of the unknown

$$\tau = \tan \frac{\gamma_2}{2}$$

This process of elimination and reduction is carried out in Sec. 11-3 and yields a fourth-degree algebraic equation

$$\tau^4 + b_3\tau^3 + b_2\tau^2 + b_1\tau + b_0 = 0 \qquad (11\text{-}18)$$

whose coefficients b_m $(m = 0, 1, 2, 3)$ are real and may be evaluated in terms of the coordinates of the accuracy points and the corresponding crank rotation. The solution of this quartic, which must be carried out by iteration, yields *zero*, *two*, or *four* real roots, from which zero, two, or four values of ν_2 may be obtained; these values are denoted as ν_{2k} with $k = 1, 2$ or $k = 1, 2, 3, 4$. The corresponding values of ν_3, ν_4, ν_5 may then be obtained from equations derived in the course of the elimination (see Sec. 11-3).

When one set of solutions $\nu_{2k}, \ldots, \nu_{5k}$ is substituted for $\nu_2, \ldots, \nu_5$ in the determinants of the second pair of compatibility equations (11-17), the problem appears identical to that of the first pair. The complex numbers $\lambda_2, \ldots, \lambda_5$ must now be replaced by $\nu_{2k}, \ldots, \nu_{5k}$, and a similar process of elimination and reduction would obviously yield a fourth-degree equation similar to (11-18), so that four roots may be expected at most. These roots, however, may be found more directly.

Let $\nu_{2h}, \ldots, \nu_{5h}$ be a set of solutions of the first pair of compatibility equations different from $\nu_{2k}, \ldots, \nu_{5k}$. Substituting these values for $\mu_2, \ldots, \mu_5$ in the determinants of (11-17) yields

$$\begin{vmatrix} \nu_{2h} - 1 & \nu_{2k} - 1 & \delta_2 \\ \nu_{3h} - 1 & \nu_{3k} - 1 & \delta_3 \\ \nu_{4h} - 1 & \nu_{4k} - 1 & \delta_4 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} \nu_{2h} - 1 & \nu_{2k} - 1 & \delta_2 \\ \nu_{3h} - 1 & \nu_{3k} - 1 & \delta_3 \\ \nu_{5h} - 1 & \nu_{5k} - 1 & \delta_5 \end{vmatrix} \qquad (11\text{-}19)$$

We shall show that, under general conditions, the above determinants are zero. With both sets $\nu_{2h}, \ldots, \nu_{5h}$ and $\nu_{2k}, \ldots, \nu_{5k}$ satisfying the first pair of compatibility equations, one set may then be used as solution of the first pair (11-16) and the other as solution of the second pair (11-17).

In order to establish that the first determinant (11-19) is zero, it will be convenient to consider the compatibility of a system of four equations

$$\begin{aligned} \delta_2 u + \delta_3 v &= \delta_4 \\ (\lambda_2 - 1)u + (\lambda_3 - 1)v &= \lambda_4 - 1 \\ (\nu_{2k} - 1)u + (\nu_{3k} - 1)v &= \nu_{4k} - 1 \\ (\nu_{2h} - 1)u + (\nu_{3h} - 1)v &= \nu_{4h} - 1 \end{aligned} \qquad (11\text{-}20)$$

with two unknowns u and v. On the assumption that

$$D_3 = \begin{vmatrix} \delta_2 & \delta_3 \\ \lambda_2 - 1 & \lambda_3 - 1 \end{vmatrix} \neq 0$$

this determinant may be chosen as principal determinant of the system, and the characteristic determinants

$$\begin{vmatrix} \delta_2 & \delta_3 & \delta_4 \\ \lambda_2 - 1 & \lambda_3 - 1 & \lambda_4 - 1 \\ \nu_{2k} - 1 & \nu_{3k} - 1 & \nu_{4k} - 1 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} \delta_2 & \delta_3 & \delta_4 \\ \lambda_2 - 1 & \lambda_3 - 1 & \lambda_4 - 1 \\ \nu_{2h} - 1 & \nu_{3h} - 1 & \nu_{4h} - 1 \end{vmatrix}$$

are zero since the sets $\nu_{2k}, \ldots, \nu_{5k}$ and $\nu_{2h}, \ldots, \nu_{5h}$ satisfy the first pair of compatibility equations (11-16). The system (11-20) is therefore compatible. However, the second-order determinant

$$D_2 = \begin{vmatrix} \mu_2 - 1 & \nu_2 - 1 \\ \mu_3 - 1 & \nu_3 - 1 \end{vmatrix} = \begin{vmatrix} \nu_{2h} - 1 & \nu_{2k} - 1 \\ \nu_{3h} - 1 & \nu_{3k} - 1 \end{vmatrix} = \begin{vmatrix} \nu_{2h} - 1 & \nu_{3h} - 1 \\ \nu_{2k} - 1 & \nu_{3k} - 1 \end{vmatrix}$$

is also different from zero. Upon choosing D_2 as principal determinant of (11-20) the characteristic determinants

$$\begin{vmatrix} \nu_{2h} - 1 & \nu_{3h} - 1 & \nu_{4h} - 1 \\ \nu_{2k} - 1 & \nu_{3k} - 1 & \nu_{4k} - 1 \\ \delta_2 & \delta_3 & \delta_4 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} \nu_{2h} - 1 & \nu_{3h} - 1 & \nu_{4h} - 1 \\ \nu_{2k} - 1 & \nu_{3k} - 1 & \nu_{4k} - 1 \\ \lambda_2 - 1 & \lambda_3 - 1 & \lambda_4 - 1 \end{vmatrix}$$

must be zero, since the system has already been shown to be compatible. But the first determinant above is identical to the first determinant (11-19), which must therefore also be zero. In similar manner, one could show that the second determinant (11-19) is also zero.

If the set $\nu_{2h}, \ldots, \nu_{5h}$ were identical to $\nu_{2k}, \ldots, \nu_{5k}$, the determinants (11-19) would also be zero because of identical columns, in which case, however, the system of Eqs. (11-15) would be impossible. Therefore, every set of solutions of the first pair of compatibility equations is also a set of solutions for the second pair. The sets used as solution of the first and second pairs of equations must not, however, be identical.

In summary, three cases may be considered, depending on the number of real roots in Eq. (11-18):

1. *No* real root—the synthesis problem has no solution.

2. *Two* real roots—there are two sets of solutions to the first pair of compatibility equations, $\nu_{2k}, \ldots, \nu_{5k}$, with $k = 1, 2$. The synthesis problem has two solutions, denoted as (1, 2) and (2, 1), where the first number between parentheses denotes the value of k defining the set of solutions used for the first pair of compatibility equations and the second number the value of h defining the set used for the second pair of compatibility equations.

3. *Four* real roots—there are four sets of solutions to the first pair of compatibility equations, $\nu_{2k}, \ldots, \nu_{5k}$, with $k = 1, 2, 3, 4$. The synthesis problem has 12 solutions; with the above notation, these solutions may be denoted as (1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3).

Example[1] Determine the dimensions of a four-bar linkage to generate a curve passing through the five points shown in Fig. 11-8, with the successive crank rotations indicated. Note that points C_2, C_3, C_4,

[1] This example is taken from Freudenstein and Sandor.

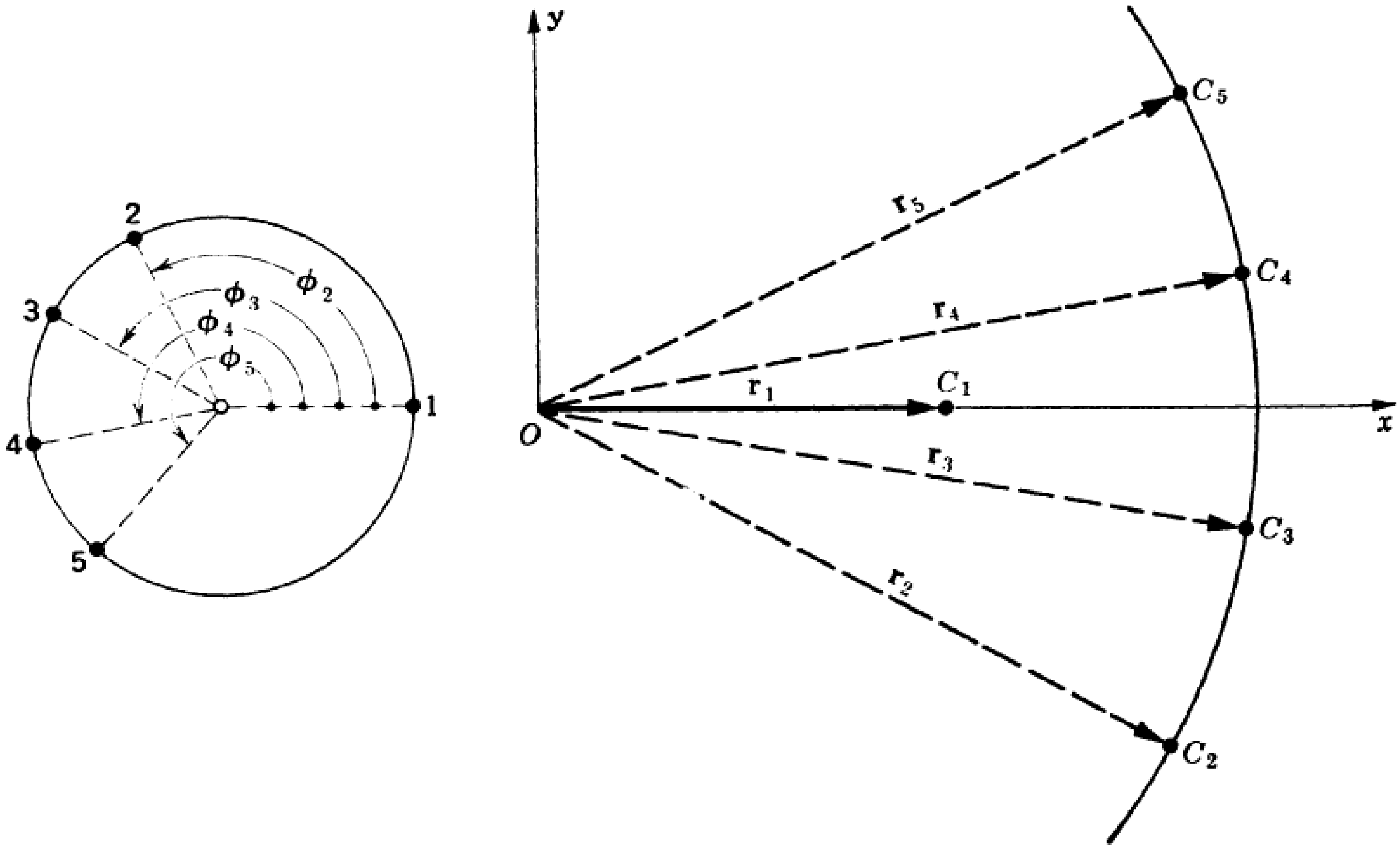


FIGURE 11-8 Example of coupler-curve synthesis, specification of crank rotations, and coupler-point positions.

and C_5 lie on a circle centered at the origin of the coordinate system; the desired coupler curve should approximate the circle as closely as possible between these points. With such specifications, the radius vectors of the five points are defined by their magnitudes and angles r_j and θ_j ($j = 1, 2, 3, 4, 5$), and the crank rotations are defined by the angles ϕ_j ($j = 2, 3, 4, 5$) (see Table 11-2).

Table 11-2 SPECIFICATION OF FIVE ACCURACY POINTS FOR COUPLER–CURVE SYNTHESIS

POSITION	ϕ_j, DEG	$r_j = \|\mathbf{r}_j\|$, IN.	θ_j, DEG
1	.....	1.0	0
2	117.0	1.740	−29.50
3	150.0	1.740	−10.70
4	191.0	1.740	10.30
5	228.0	1.740	25.90