10

ALGEBRAIC METHODS
OF SYNTHESIS USING
DISPLACEMENT EQUATIONS

10-1 DISPLACEMENT EQUATION OF THE
FOUR-BAR LINKAGE

Consider a planar four-bar linkage $O_AABO_B$ (Fig. 10-1). This
linkage is characterized by having four revolutes with parallel
axes, the distances between successive axes being the parameters
$a_1, a_2, a_3, a_4$. The synthesis of four-bar linkages, or the deter-
mination of the four parameters that will yield an approximation
to a desired function between the input (crank) and output (fol-
lower) angles, has been approached in the last chapters by geo-
metric methods. In this chapter, algebraic methods for the
synthesis of four-bar linkages as well as other planar mechanisms
will be considered. Such methods of synthesis are based on dis-
placement equations, i.e., equations relating the input and output
variables of a mechanism in terms of its fixed parameters.

The displacement equation of the four-bar linkage may be
obtained by considering a rectangular-coordinate system $O_{axy}$
(Fig. 10-1) with respect to which the coordinates of $A$ and $B$
may be written as follows:
For $A$:
\[ x_2 = a_1 \cos \phi \]
\[ y_2 = a_1 \sin \phi \]

For $B$:
\[ x_3 = -a_4 + a_3 \cos \psi \]
\[ y_3 = a_3 \sin \psi \]

Since the distance $AB$ is fixed and equal to $a_2$, application of Pythagoras’ theorem yields
\[ (x_2 - x_3)^2 + (y_2 - y_3)^2 = a_2^2 \]
or
\[ (a_1 \cos \phi + a_4 - a_3 \cos \psi)^2 - (a_1 \sin \phi - a_3 \sin \psi)^2 = a_2^2 \]

After trigonometric simplifications this may be written
\[ A \sin \psi + B \cos \psi = C \quad (10-1) \]

where\(^1\)
\[ A = \sin \phi \quad B = \frac{a_4}{a_1} + \cos \phi \quad C = \frac{a_4}{a_3} \cos \phi + \frac{a_1^2 - a_3^2 + a_2^2 + a_4^2}{2a_1a_3} \]

Equation (10-1) may be solved for a displacement analysis of the four-bar linkage; that is, $\psi$ is found explicitly as a function of $\phi$ and the parameters $a_1$, $a_2$, $a_3$, $a_4$. Such a solution is obtained by expressing $\sin \psi$ and $\cos \psi$ in terms of $\tan (\psi/2)$,
\[ \sin \psi = \frac{2 \tan (\psi/2)}{1 + \tan^2 (\psi/2)} \quad \cos \psi = \frac{1 - \tan^2 (\psi/2)}{1 + \tan^2 (\psi/2)} \]

and substituting those values in Eq. (10-1) to get
\[ 2A \tan \frac{\psi}{2} + B \left( 1 - \tan^2 \frac{\psi}{2} \right) = C \left( 1 + \tan^2 \frac{\psi}{2} \right) \]

or
\[ (B + C) \tan^2 \frac{\psi}{2} - 2A \tan \frac{\psi}{2} - B + C = 0 \]

from which
\[ \tan \frac{\psi}{2} = \frac{A \pm \sqrt{A^2 + B^2 - C^2}}{B + C} \]

For each value of $\phi$ the quantities $A$, $B$, $C$ may be obtained and
\(^1\) Do not confuse the quantities $A$, $B$, and $C$ with the points $A$ and $B$ (Fig. 10-1).

---

![Figure 10-1 Planar four-bar linkage; coordinates of $A$ and $B$.](image-url)
two distinct values of \( \psi \) found as

\[
\begin{align*}
\psi^+ &= 2 \arctan \frac{A + \sqrt{A^2 + B^2 - C^2}}{B + C} \\
\psi^- &= 2 \arctan \frac{A - \sqrt{A^2 + B^2 - C^2}}{B + C}
\end{align*}
\] (10-2)

These two values correspond to the two ways in which a four-bar linkage may be closed (Fig. 10-2).

10-2 CRANK AND FOLLOWER SYNTHESIS: THREE ACCURACY POINTS

Consider the problem of designing a planar four-bar linkage such that to three given positions of the crank, defined by angles \( \phi_1, \phi_2, \) and \( \phi_3, \) there correspond three prescribed positions of the follower, \( \psi_1, \psi_2, \) and \( \psi_3. \) The solution consists in finding the proper values of \( a_1, a_2, a_3, \) and \( a_4 \) for three related pairs \( (\phi_1, \psi_1), (\phi_2, \psi_2), \) and \( (\phi_3, \psi_3). \) The procedure is based on the displacement equation\(^1\)

\[
K_1 \cos \phi - K_2 \cos \psi + K_3 = \cos (\phi - \psi)
\] (10-3)

with

\[
K_1 = \frac{a_4}{a_5} \quad K_2 = \frac{a_4}{a_1} \quad K_3 = \frac{a_1^2 - a_2^2 + a_3^2 + a_4^2}{2a_1a_2}
\]

This equation was deduced from Eq. (10-1) by rearranging the terms. When written for three pairs of values, \( (\phi_1, \psi_1), (\phi_2, \psi_2), (\phi_3, \psi_3), \) this equation yields a system of three equations linear with respect to \( K_1, K_2, K_3, \)

\[
\begin{align*}
K_1 \cos \phi_1 - K_2 \cos \psi_1 + K_3 &= \cos (\phi_1 - \psi_1) \\
K_1 \cos \phi_2 - K_2 \cos \psi_2 + K_3 &= \cos (\phi_2 - \psi_2) \\
K_1 \cos \phi_3 - K_2 \cos \psi_3 + K_3 &= \cos (\phi_3 - \psi_3)
\end{align*}
\]

\(^1\)This is also known as the Freudenstein equation (see first reference in Bibliography at the end of this chapter).
Tedious third-order determinants may be avoided by first subtracting the second and third equations from the first, thus eliminating \( K_3 \),

\[
\begin{align*}
K_1(\cos \phi_1 - \cos \phi_2) - K_2(\cos \psi_1 - \cos \psi_2) &= \cos (\phi_1 - \psi_1) - \cos (\phi_2 - \psi_2) \\
K_1(\cos \phi_1 - \cos \phi_3) - K_2(\cos \psi_1 - \cos \psi_3) &= \cos (\phi_1 - \psi_1) - \cos (\phi_3 - \psi_3)
\end{align*}
\]

and solving the resulting system of two equations with two unknowns; thus,

\[
K_1 = \frac{w_2w_6 - w_3w_5}{w_2w_4 - w_1w_5} \quad K_2 = \frac{w_3w_6 - w_4w_4}{w_2w_4 - w_1w_5}
\]

in which

\[
\begin{align*}
w_1 &= \cos \phi_1 - \cos \phi_2 \\
w_2 &= \cos \psi_1 - \cos \psi_2 \\
w_3 &= \cos (\phi_1 - \psi_1) - \cos (\phi_2 - \psi_2) \\
w_4 &= \cos \phi_1 - \cos \phi_3 \\
w_5 &= \cos \psi_1 - \cos \psi_3 \\
w_6 &= \cos (\phi_1 - \psi_1) - \cos (\phi_3 - \psi_3)
\end{align*}
\]

Substituting values of \( K_1 \) and \( K_2 \) into one of the three original equations yields \( K_3 \) as

\[
K_3 = \cos (\phi_i - \psi_i) - K_1 \cos \phi_i + K_2 \cos \psi_i \quad i = 1, 2, \text{ or } 3
\]

With the values of \( K_1, K_2, \) and \( K_3 \) known, the parameters of the linkage may be found from the relations

\[
a_1 = \frac{a_1}{K_2} \quad a_3 = \frac{a_3}{K_1} \quad a_2 = \sqrt{a_3^2 + a_4^2 + a_4^2 - 2a_3a_4K_3}
\]

The parameter \( a_4 \) may be given a positive but arbitrary value, usually taken as unity. This parameter merely determines the size of the linkage and has no effect on the angular relationships.

10-3 **EXAMPLES: FOUR-BAR FUNCTION GENERATORS WITH THREE ACCURACY POINTS**

The design of four-bar function generators, already carried out by geometric methods in Secs. 8-4 and 8-5, is reconsidered here as an application of the three-accuracy-point synthesis developed in the last section.

**Example 1** The function \( y = \log x \) is to be generated in the interval \( 1 \leq x \leq 2 \) by means of a four-bar linkage \( O_1A_1B_1O_2 \) (Fig. 10-3). The basic elements of the problem are here the same as in Sec. 8-4. The variables \( x \) and \( y \) are represented, respectively, by the crank and follower
angles $\phi$ and $\psi$ through the relations

\[
\frac{\phi - \phi_s}{\Delta \phi} = \frac{x - x_s}{\Delta x} \quad \frac{\psi - \psi_s}{\Delta \psi} = \frac{y - y_s}{\Delta y}
\]

The reader is referred to Sec. 8-4 for the details of the formulation of the problem and the definitions of the symbols used. Three accuracy points are taken in the interval $1 \leq x \leq 2$ with Chebyshev spacing, whence the corresponding values of the variables $x$ and $y$ are

\[
x_1 = 1.067 \quad y_1 = 0.0282 \\
x_2 = 1.5 \quad y_2 = 0.1761 \\
x_3 = 1.933 \quad y_3 = 0.2862
\]

The ranges of variation of $\phi$ and $\psi$ must be selected. They are chosen as $\Delta \phi = \Delta \psi = 60^\circ$. The rotations of the crank and follower from the position corresponding to the first accuracy point to the positions corresponding to the other two are, with the computation carried to $\frac{1}{100}$,

\[
\phi_2 - \phi_1 = \frac{x_2 - x_1}{x_f - x_s} \Delta \phi = 26.0^\circ \quad \psi_2 - \psi_1 = \frac{y_2 - y_1}{y_f - y_s} \Delta \psi = 29.4^\circ \\
\phi_3 - \phi_1 = \frac{x_3 - x_1}{x_f - x_s} \Delta \phi = 52.0^\circ \quad \psi_3 - \psi_1 = \frac{y_3 - y_1}{y_f - y_s} \Delta \psi = 51.4^\circ
\]

With the present method, the angles $\phi_1$ and $\psi_1$, crank and follower positions corresponding to the first accuracy point, must also be selected at the start. Choosing $\phi_1 = 0$ and $\psi_1 = 0$ yields

\[
\phi_2 = 26.0^\circ \quad \psi_2 = 29.4^\circ \\
\phi_3 = 52.0^\circ \quad \psi_3 = 51.4^\circ
\]
from which \( w_1 = 0.1011 \quad w_2 = 0.1285 \quad w_3 = 0.0017 \)
\( w_4 = 0.3839 \quad w_5 = 0.3766 \quad w_6 = 0 \)
giving \( K_1 = -0.05777 \quad K_2 = -0.05900 \quad K_3 = 0.99877 \)

With the frame \( a_4 = 1 \) unit of length, the other three parameters of the linkage are found as
\[
a_1 = -16.95 \quad a_2 = 1.36 \quad a_3 = -17.31
\]

This linkage, with two long links (crank \( a_1 = 16.95 \), follower \( a_3 = 17.31 \)) and two relatively short links (frame \( a_4 = 1.00 \), coupler \( a_2 = 1.36 \)), has poor force-transmission qualities and is not an acceptable solution. In point of fact, application of the Grashof criterion shows this linkage to have change points; on second thought, this was inevitable, since both crank and follower had starting angles of 0°. The unsatisfactory solution was compounded from unhappy choices of arbitrary values—starting angles and ranges of motion.

If the idea of the spread associated with the 60° ranges seems desirable and this feature is to be retained, only one alternative exists, viz., different starting positions for \( \phi \) and \( \psi \).

A second attempt, in which \( \phi_1 = 45^\circ \) (\( \phi_2 = 71^\circ \), \( \phi_3 = 97^\circ \)) and \( \psi_1 = 0^\circ \) (\( \psi_2 = 29.5^\circ \), \( \psi_3 = 51.4^\circ \)) were assumed, with \( a_4 = 1.0 \), yielded \( a_1 = -1.031 \), \( a_2 = 2.682 \), \( a_3 = -2.310 \). These linkage proportions are favorable to force transmission, and the design may be considered as acceptable, if it is recognized that it is a double rocker. The new linkage, drawn in position 1, is shown in Fig. 10-4. The negative signs for \( a_1 \) and

![Figure 10-4](image_url)

**Figure 10-4** Example 1, function generator \( y = \log x \), \( 1 \leq x \leq 2 \), with three accuracy points, second attempt.
$a_3$ are interpreted by considering $O_A A$ and $O_B B$ as vectors: the angles $\phi$ and $\psi$ define their direction; the parameters $a_1$ and $a_3$ define their magnitudes and the sense in which they are to be laid off. A graphical check of this linkage for the three accuracy points shows that no large error is present. To determine the structural error accurately, an analysis must be carried out by using Eqs. (10-2) developed in Sec. 10-1. The results of this analysis for values of $\phi$ in the interval $\phi_s \leq \phi \leq \phi_f$ at $6^\circ$ intervals are summarized in Table 10-1.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\phi$, DEG</th>
<th>$\psi$, DEG</th>
<th>log $x$</th>
<th>$y_{\text{mech}}$</th>
<th>$y_{\text{mech}} - \log x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>41.0</td>
<td>-6.1</td>
<td>0</td>
<td>-0.025</td>
<td>-0.025</td>
</tr>
<tr>
<td>1.1</td>
<td>47.0</td>
<td>2.8</td>
<td>0.041</td>
<td>0.042</td>
<td>0.001</td>
</tr>
<tr>
<td>1.2</td>
<td>53.0</td>
<td>10.4</td>
<td>0.079</td>
<td>0.080</td>
<td>0.001</td>
</tr>
<tr>
<td>1.3</td>
<td>59.0</td>
<td>17.3</td>
<td>0.114</td>
<td>0.115</td>
<td>0.001</td>
</tr>
<tr>
<td>1.4</td>
<td>65.0</td>
<td>23.5</td>
<td>0.146</td>
<td>0.146</td>
<td>0</td>
</tr>
<tr>
<td>1.5</td>
<td>71.0</td>
<td>29.4</td>
<td>0.176</td>
<td>0.176</td>
<td>0</td>
</tr>
<tr>
<td>1.6</td>
<td>77.0</td>
<td>34.9</td>
<td>0.204</td>
<td>0.203</td>
<td>-0.001</td>
</tr>
<tr>
<td>1.7</td>
<td>83.0</td>
<td>40.1</td>
<td>0.230</td>
<td>0.229</td>
<td>-0.001</td>
</tr>
<tr>
<td>1.8</td>
<td>89.0</td>
<td>45.1</td>
<td>0.255</td>
<td>0.254</td>
<td>-0.001</td>
</tr>
<tr>
<td>1.9</td>
<td>95.0</td>
<td>49.9</td>
<td>0.279</td>
<td>0.278</td>
<td>-0.001</td>
</tr>
<tr>
<td>2.0</td>
<td>101.0</td>
<td>54.5</td>
<td>0.301</td>
<td>0.302</td>
<td>0.001</td>
</tr>
</tbody>
</table>

By taking $\psi = \psi^+$, the structural error, i.e., the difference between the values of $y_{\text{mech}}$ given by the linkage and the corresponding values of log $x$, is shown in the last column. As expected, this structural error vanishes at the accuracy points. The maximum structural error, occurring at $x = 1.0$, is $\epsilon = -0.025$, or 8.3 percent of the range of variation of $y$.

With algebraic methods of synthesis and analysis, all quantities may be calculated to any desired degree of accuracy, which means the elimination of graphical error. Structural error remains, as does the mechanical error deriving from machining tolerances and deformations of links. The evaluation of this mechanical error will be considered in a later section.

**Example 2** A four-bar linkage is to generate the function $y = 1/x$ over the interval $1 \leq x \leq 2$. The ranges of variation are to be $\Delta \phi = 90^\circ$ and $\Delta \psi = 90^\circ$ with three accuracy points having Chebyshev spacing. The accuracy points are then

$$x_1 = 1.067 \quad x_2 = 1.5 \quad x_3 = 1.933$$
with associated functional values of

\[ y_1 = 0.9372 \quad y_2 = 0.6666 \quad y_3 = 0.5173 \]

Taking \( \phi_1 = 45^\circ, \psi_1 = 0 \) yields

\[
\phi_2 = \phi_1 + \frac{x_2 - x_1}{x_f - x_s} \Delta \phi = 84.0^\circ \\
\psi_2 = \psi_1 + \frac{y_2 - y_1}{y_f - y_s} \Delta \psi = -48.7^\circ \\
\phi_3 = \phi_1 + \frac{x_3 - x_1}{x_f - x_s} \Delta \phi = 123.0^\circ \\
\psi_3 = \psi_1 + \frac{y_3 - y_1}{y_f - y_s} \Delta \psi = -75.6^\circ
\]

from which the linkage parameters are found as

\[ a_1 = 0.036 \quad a_2 = 0.970 \quad a_3 = 0.056 \quad \text{with } a_4 = 1.0 \]

In this linkage the crank and follower lengths are very small compared with the frame and coupler lengths, and another "try" is indicated. We hopefylly choose \( \phi_1 = 45^\circ, \psi_1 = 90^\circ \). The crank angles \( \phi \) corresponding

---

**Figure 10-5** Example 2, function generator \( y = 1/x, 1 \leq x \leq 2 \), with three accuracy points, second attempt.
to the accuracy points remain as above, but the follower angles become

$$\psi_1 = 90^\circ \quad \psi_2 = 41.3^\circ \quad \psi_3 = 14.4^\circ$$

With these new specifications the linkage parameters are found as

$$a_1 = -0.547 \quad a_2 = 1.035 \quad a_3 = 0.447 \quad \text{when } a_4 = 1.0$$

This new linkage is shown in Fig. 10-5. A displacement analysis similar to that described in the last example shows that the maximum structural error is $\epsilon = 0.015$, that is, 3 percent of the range of variation of $y$.

10-4 CRANK AND FOLLOWER SYNTHESIS: ANGULAR VELOCITIES AND ACCELERATIONS

The design of a planar four-bar linkage having prescribed angular velocities and angular accelerations of the crank and follower for a desired phase, i.e., for specified values of $\phi$ and $\psi$, may be carried out in a fashion similar to the foregoing. If, in the four-bar linkage of Fig. 10-6, the crank angle is $\phi$, the angular velocity and angular acceleration of the crank are defined by

$$\omega_\phi = \frac{d\phi}{dt} \quad \text{and} \quad \alpha_\phi = \frac{d^2\phi}{dt^2}$$

With a follower angle $\psi$, the angular velocity and angular acceleration of the follower become

$$\omega_\psi = \frac{d\psi}{dt} \quad \text{and} \quad \alpha_\psi = \frac{d^2\psi}{dt^2}$$

A linkage may be designed such that, when the crank has a specified position, angular velocity, and acceleration, the follower will also have a specified position, angular velocity, and acceleration. In other words, it is necessary to determine the parameters $a_1$, $a_2$, $a_3$, $a_4$ such that a given set of values $\phi$, $\omega_\phi$, $\alpha_\phi$ will give rise to desired values of $\psi$, $\omega_\psi$, $\alpha_\psi$.

This problem may be solved by taking the first and second time
derivatives of Eq. (10-3),

\[
K_1 \cos \phi - K_2 \cos \psi + K_3 = \cos (\phi - \psi)
\]

\[
K_1 \omega_\phi \sin \phi - K_2 \omega_\psi \sin \psi = (\omega_\phi - \omega_\psi) \sin (\phi - \psi)
\]

\[
K_1(\alpha_\phi \sin \phi + \omega_\phi^2 \cos \phi) - K_2(\alpha_\psi \sin \psi + \omega_\psi^2 \cos \psi)
= (\alpha_\phi - \alpha_\psi) \sin (\phi - \psi) + (\omega_\phi - \omega_\psi)^2 \cos (\phi - \psi)
\]

Solving the last two equations of this system for \( K_1 \) and \( K_2 \) and substituting the values obtained in the first equation to get \( K_3 \) yields

\[
K_1 = \frac{w_3w_6 - w_3w_5}{w_3w_4 - w_1w_5} \quad K_2 = \frac{w_1w_6 - w_3w_4}{w_2w_4 - w_1w_5}
\]

where

\[
w_1 = \omega_\phi \sin \phi \]

\[
w_2 = \omega_\psi \sin \psi \]

\[
w_3 = (\omega_\phi - \omega_\psi) \sin (\phi - \psi) \]

\[
w_4 = \alpha_\phi \sin \phi + \omega_\phi^2 \cos \phi \]

\[
w_5 = \alpha_\psi \sin \psi + \omega_\psi^2 \cos \psi \]

\[
w_6 = (\alpha_\phi - \alpha_\psi) \sin (\phi - \psi) + (\omega_\phi - \omega_\psi)^2 \cos (\phi - \psi)
\]

and

\[
K_3 = \cos (\phi - \psi) - K_1 \cos \phi + K_2 \cos \psi
\]

As in the case of three-accuracy-point synthesis, the parameters of the linkage are given by

\[
a_1 = \frac{a_4}{K_2} \quad a_3 = \frac{a_1}{K_1} \quad a_2 = \sqrt{a_1^2 + a_3^2 + a_4^2 - 2a_1a_3K_3}
\]

**Example** Design a four-bar linkage meeting the following specifications:

\[
\phi = -144^\circ \quad \omega_\phi = -3.0 \text{ rad/sec} \quad \alpha_\phi = 0
\]

\[
\psi = 65^\circ \quad \omega_\psi = 8.0 \text{ rad/sec} \quad \alpha_\psi = 0
\]

Taking \( a_4 = 3.760 \), * the remaining parameters of the required linkage are

* This value of \( a_4 \) was chosen to allow a comparison with a solution gained from complex numbers (Sec. 11-1).
found as

\[ a_1 = 1.676 \quad a_2 = 2.640 \quad a_3 = 0.606 \]

The linkage is shown in Fig. 10-7.

10-5 GENERALIZATION OF THE SYNTHESIS METHOD
BY LINEAR EQUATIONS

The foregoing methods of synthesis of the four-bar linkage in terms of linear equations only were made possible by the form of the displacement equation [Eq. (10-3)]. In this, the three coefficients \( K_1 \), \( K_2 \), and \( K_3 \) are (1) functions of three design parameters \( a_1/a_1 \), \( a_2/a_4 \), \( a_3/a_4 \), and (2) independent of the input and output variables, viz., the crank and follower angles \( \phi \) and \( \psi \). It is therefore possible to generalize this method to other linkages for which the displacement equation may be written in a form having essentially the same properties. Consider a general linkage in which the number of design parameters is \( n \) and for which \( \phi \) and \( \psi \) are the input and output variables, analogous to the crank and follower angles in the case of the four-bar linkage. If the displacement equation of the linkage is written in the form

\[ \sum_{i=1}^{n} K_i(\text{n design parameters})G_i(\phi, \psi) = F(\phi, \psi) \]

then, writing this equation for \( n \) pairs of values \( (\phi_j, \psi_j), j = 1, 2, \ldots, n \), there will be \( n \) equations linear in \( K_1, K_2, \ldots, K_n \),

\[
\begin{align*}
K_1G_1(\phi_1, \psi_1) + K_2G_2(\phi_1, \psi_1) + \cdots + K_nG_n(\phi_1, \psi_1) &= F(\phi_1, \psi_1) \\
K_1G_1(\phi_2, \psi_2) + K_2G_2(\phi_2, \psi_2) + \cdots + K_nG_n(\phi_2, \psi_2) &= F(\phi_2, \psi_2) \\
&\vdots \\
K_1G_1(\phi_n, \psi_n) + K_2G_2(\phi_n, \psi_n) + \cdots + K_nG_n(\phi_n, \psi_n) &= F(\phi_n, \psi_n)
\end{align*}
\]

which may be solved for \( K_1, K_2, \ldots, K_n \). Since these \( K \)'s are functions of \( n \) design parameters, this leads to a set of \( n \) equations from which the \( n \) design parameters may be found.

10-6 SYNTHESIS OF THE SLIDER-CRANK MECHANISM
WITH THREE ACCURACY POINTS

The general approach to synthesis outlined in the last section will be applied here to the slider-crank mechanism (Fig. 10-8). The first step in the synthesis is to derive a displacement equation taking into account all possible parameters of the linkage. This may be achieved by writing
the coordinates of points \( A \) and \( B \) with respect to the set of axes \( O_{Axy} \) as

For \( A \):
\[
\begin{align*}
    x_A &= a_1 \cos \phi \\
    y_A &= a_1 \sin \phi
\end{align*}
\]

For \( B \):
\[
\begin{align*}
    x_B &= s \\
    y_B &= a_3
\end{align*}
\]

and then expressing the distance \( AB = a_2 \) as
\[
(AB)^2 = (x_B - x_A)^2 + (y_B - y_A)^2
\]
or
\[
a_2^2 = (s - a_1 \cos \phi)^2 + (a_3 - a_1 \sin \phi)^2
\]

This, after manipulation and reduction by means of trigonometric identities, becomes

\[
2a_1 s \cos \phi - 2a_1 a_3 \sin \phi = (a_1^2 - a_2^2 + a_3^2) = s^2 \quad (10-4)
\]

To carry out a three-point synthesis relating the crank position \( \phi \) to the slider position \( s \), three coefficients \( K_1, K_2, K_3 \) must be defined in terms of the three parameters \( a_1, a_2, a_3 \) of the linkage. On setting
\[
K_1 = 2a_1, \quad K_2 = 2a_1 a_3, \quad K_3 = a_1^2 - a_2^2 + a_3^2
\]

Eq. (10-4) takes the form
\[
K_1 s \cos \phi + K_2 \sin \phi - K_3 = s^2 \quad (10-5)
\]

which satisfies the conditions of the last section. With the notation of the last section,
\[
G_1 = s \cos \phi, \quad G_2 = \sin \phi, \quad G_3 = -1, \quad F = s^2
\]

are recognized as functions of the input and output variables \( \phi \) and \( s \) but independent of the design parameters \( a \). The coefficients \( K \), on the other hand, are functions of the design parameters while independent of the

\textbf{Figure 10-8} Slider-crank mechanism showing parameters and variables used in synthesis.
input and output variables. Writing Eq. (10-5) for three pairs of values
(\(\phi_1, s_1\)), (\(\phi_2, s_2\)), and (\(\phi_3, s_3\)) yields the system

\[
\begin{align*}
K_1s_1 \cos \phi_1 + K_2 \sin \phi_1 - K_3 &= s_1^2 \\
K_1s_2 \cos \phi_2 + K_2 \sin \phi_2 - K_3 &= s_2^2 \\
K_1s_3 \cos \phi_3 + K_2 \sin \phi_3 - K_3 &= s_3^2 \\
K_1 &= \frac{w_2w_6 - w_3w_5}{w_2w_1 - w_1w_5} \\
K_2 &= \frac{w_3w_4 - w_1w_6}{w_2w_1 - w_1w_5}
\end{align*}
\]

\[
\begin{align*}
w_1 &= s_1 \cos \phi_1 - s_2 \cos \phi_2 \\
w_2 &= s_1 \sin \phi_1 - \sin \phi_2 \\
w_3 &= s_1^2 - s_2^2 \\
w_4 &= s_1 \cos \phi_1 - s_3 \cos \phi_3 \\
w_5 &= \sin \phi_1 - \sin \phi_3 \\
w_6 &= s_1^2 - s_3^2
\end{align*}
\]

and \(K_3\) is conveniently given by

\[
K_3 = -s_i + K_1 s_i \cos \phi_i + K_2 \sin \phi_i \quad i = 1, 2, \text{ or } 3
\]

The parameters of the linkage now follow as

\[
a_1 = \frac{K_1}{2} \quad a_3 = \frac{K_2}{2a_1} \quad a_2 = \sqrt{a_1^2 + a_3^2 - K_3}
\]

**Example** Design a slider-crank mechanism in which the slider displacement is proportional to the square of the crank rotation, or

\[
\frac{s - s_s}{s_f - s_s} = \left(\frac{\phi - \phi_s}{\phi_f - \phi_s}\right)^2 \quad (10-6)
\]

where \(\Delta s = s_f - s_s\) is the total displacement of the slider corresponding to the crank rotation \(\Delta \phi = \phi_f - \phi_s\). The starting and final values \(\phi_s\), \(s_s\), and \(\phi_f, s_f\) are unspecified and may be chosen by the designer. The following values are assumed for the present example:

\[
\begin{align*}
\phi_s &= 45^\circ & s_s &= 8 \text{ in.} \\
\phi_f &= 105^\circ & s_f &= 12 \text{ in.}
\end{align*}
\]

The range of variation of the crank is therefore \(\Delta \phi = \phi_f - \phi_s = 60^\circ\), and the corresponding displacement of the slider is \(\Delta s = s_f - s_s = 4 \text{ in.}\). Three accuracy points are chosen with Chebyshev spacing and yield

\[
\phi_1 = 49^\circ \quad \phi_2 = 75^\circ \quad \phi_3 = 101^\circ
\]

The corresponding values of \(s\) are deduced from Eq. (10-6) as

\[
\begin{align*}
s_1 &= s_s + (s_f - s_s) \left(\frac{\phi_1 - \phi_s}{\phi_f - \phi_s}\right)^2 = 8.02 \text{ in.} \\
s_2 &= 9.00 \text{ in.} \\
s_3 &= 11.48 \text{ in.}
\end{align*}
\]
The values of the coefficients \( w \) follow as
\[
\begin{align*}
    w_1 &= 2.93  \\
    w_2 &= -0.211  \\
    w_3 &= -16.8  \\
    w_4 &= 7.46  \\
    w_5 &= -0.227  \\
    w_6 &= -67.8
\end{align*}
\]
from which \( K_1 = -11.7 \), \( K_2 = -84.0 \), \( K_3 = -189.1 \).

The dimensions of the linkage are found to be
\[
\begin{align*}
    a_1 &= -5.85 \text{ in.}  \\
    a_2 &= 16.55 \text{ in.}  \\
    a_3 &= 7.18 \text{ in.}
\end{align*}
\]

This linkage is shown in Fig. 10-9.

10-7 SYNTHESIS OF THE SLIDER-CRANK MECHANISM WITH FOUR ACCURACY POINTS

To extend the three-accuracy-point synthesis of the slider-crank mechanism to a higher number of accuracy points, additional design parameters must be taken into account. In the case of four accuracy points, either the crank position \( \phi_1 \) or the slider position \( s_1 \) corresponding to the first accuracy point must be included with the link lengths \( a_1, a_2, \) and \( a_3 \) as a fourth design parameter. The slider position \( s_1 \) will be taken in this section as a design parameter, leaving \( \phi_1 \) to be chosen by the designer to meet additional requirements. The slider position corre-
sponding to any accuracy point may now be defined as \( s_{ij} = s_1 + s_{ij} \), where \( s_{ij} \) is the slider displacement from the position corresponding to the first accuracy point to points 1, 2, 3, and 4 (\( s_{11} \) being always zero). In these terms, the displacement equation of the mechanism becomes

\[
2a_1(s_1 + s_{1j}) \cos \phi_j + 2a_1a_3 \sin \phi_j - (a_1^2 - a_2^2 + a_3^2) = (s_1 + s_{1j})^2
\]

or

\[
2a_1s_1 \cos \phi_j + 2a_1s_{1j} \cos \phi_j + 2a_1a_3 \sin \phi_j - (a_1^2 - a_2^2 + a_3^2 + s_{1j}^2) = 2s_1s_{1j} + s_{1j}^2
\]

An attempt to apply the general approach to synthesis by linear equations to the present problem would yield four linear equations (corresponding to four accuracy points) with five unknowns,

\[
K_1 \cos \phi_j + K_2s_{1j} \cos \phi_j + K_3 \sin \phi_j - K_4 = K_5s_{1j} + s_{1j}^2
\]

\[
\begin{align*}
K_1 &= 2a_1s_1, & K_2 &= 2a_1, \\
K_3 &= 2a_1a_3, & K_4 &= a_1^2 - a_2^2 + a_3^2 + s_{1j}^2, & K_5 &= 2s_1
\end{align*}
\]

\[ j = 1, 2, 3, 4 \quad (10-7) \]

in which

\[ 2K_1 = K_2K_5 \quad \text{or} \quad 2K_1 - K_2K_5 = 0 \quad (10-8) \]

This additional relation, called the compatibility equation, must be added to the system of four linear equations (10-7) to yield a system of five equations for the five unknowns \( K_i \) with \( i = 1, 2, 3, 4, 5 \).

Since the compatibility equation (10-8) is nonlinear, the above system cannot be solved by the simple rules involving five linear equations with five unknowns. Its solution will be better understood if \( K_5 \) is denoted as \( \lambda \). The compatibility equation then becomes

\[ 2K_1 - K_2\lambda = 0 \quad (10-9) \]

and the system of equations (10-7) is rewritten as

\[
K_1 \cos \phi_j + K_2s_{1j} \cos \phi_j + K_3 \sin \phi_j - K_4 = \lambda s_{1j} + s_{1j}^2
\]

\[ j = 1, 2, 3, 4 \quad (10-10) \]

This system cannot be solved, since the value of \( \lambda \) is unknown; however, \( K_1, K_2, K_3, K_4 \) may be expressed in terms of \( \lambda \). To achieve this, consider
Table 10-2  SYNTHESIS OF SLIDER–CRANK MECHANISM, FOUR ACCURACY POINTS

Specifications:
Crank rotations $\phi_{12}$, $\phi_{13}$, $\phi_{14}$
Slider displacements $s_{12}$, $s_{13}$, $s_{14}$
Crank position corresponding to first accuracy point, $\phi_1$

Parameters:
$a_1$, $a_2$, $a_3$, $s_1$

Procedure:
1. Compute
$$\phi_2 = \phi_1 + \phi_{12} \quad \phi_3 = \phi_1 + \phi_{13} \quad \phi_4 = \phi_1 + \phi_{14}$$

2. Solve the two systems of linear equations
$$l_1(\cos \phi_1) + l_3(\sin \phi_1) - l_4 = 0$$
$$l_1(\cos \phi_2) + l_2(s_{12} \cos \phi_2) + l_3(\sin \phi_2) - l_4 = s_{12}$$
$$l_1(\cos \phi_3) + l_2(s_{13} \cos \phi_3) + l_3(\sin \phi_3) - l_4 = s_{13}$$
$$l_1(\cos \phi_4) + l_2(s_{14} \cos \phi_4) + l_3(\sin \phi_4) - l_4 = s_{14}$$

and
$$m_1(\cos \phi_1) + m_3(\sin \phi_1) - m_4 = 0$$
$$m_1(\cos \phi_2) + m_2(s_{12} \cos \phi_2) + m_3(\sin \phi_2) - m_4 = s_{12}^2$$
$$m_1(\cos \phi_3) + m_2(s_{13} \cos \phi_3) + m_3(\sin \phi_3) - m_4 = s_{13}^2$$
$$m_1(\cos \phi_4) + m_2(s_{14} \cos \phi_4) + m_3(\sin \phi_4) - m_4 = s_{14}^2$$

Note: The coefficients of the $l$'s and $m$'s are the same in both systems; only the second members differ, allowing the elimination of unknowns to be carried out in similar fashion for both systems.

3. Compute the discriminant $\Delta = (m_2 - 2l_3)^2 + 8m_1l_2$:
If $\Delta < 0$, there is no solution.

If $\Delta = 0$, $\lambda = \frac{2l_1 - m_2}{2l_2}$; solution is unique.

If $\Delta > 0$, $\lambda = \frac{2l_1 - m_2 + \sqrt{\Delta}}{2l_2}$ or $\lambda = \frac{2l_1 - m_2 - \sqrt{\Delta}}{2l_2}$; there are two solutions, one for each $\lambda$.

4. Compute for each $\lambda$:
$$K_1 = \lambda l_1 + m_1 \quad \text{and} \quad a_1 = \frac{K_2}{2}$$
$$K_2 = \lambda l_2 + m_2 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad s_1 = \frac{\lambda}{2}$$
$$K_3 = \lambda l_3 + m_3 \quad a_3 = \frac{K_3}{K_2}$$
$$K_4 = \lambda l_4 + m_4 \quad a_2 = \sqrt{a_1^2 + a_3^2 + s_1^2 - K_4}$$
the two systems of linear equations

\[ l_1 \cos \phi_j + l_2 s_{1j} \cos \phi_j + l_3 \sin \phi_j - l_4 = s_{1j} \]

and

\[ m_1 \cos \phi_j + m_2 s_{1j} \cos \phi_j + m_3 \sin \phi_j - m_4 = s_{1j}^2 \]

obtained by considering one term of the second member at a time. These systems may be solved to yield \( l_1, l_2, l_3, l_4 \) and \( m_1, m_2, m_3, m_4 \). The unknowns \( K_1, K_2, K_3, K_4 \) of the system of equations (10-10) may now be expressed by superposition as

\[ K_i = \lambda l_i + m_i \quad i = 1, 2, 3, 4 \] (10-11)

These values substituted into the compatibility equation (10-9) yield

\[ 2(\lambda l_1 + m_1) - (\lambda l_2 + m_2)\lambda = 0 \]

or

\[ l_2\lambda^2 + (m_2 - 2l_1)\lambda - 2m_1 = 0 \] (10-12)

in which \( \lambda \) is the only unknown.

When the discriminant of this equation is positive, two values of \( \lambda \) may be found. For each of these values, \( K_1, K_2, K_3, K_4 \) may be evaluated by means of Eq. (10-11), after which the parameters \( a_1, a_2, a_3 \) and the slider position \( s_i \) corresponding to the first accuracy point may be found. Thus, if the discriminant of Eq. (10-12) is positive, the problem has two solutions; if the discriminant is zero, there is a unique solution; and if the discriminant is negative the problem has no solution. The complete numerical procedure involved is summarized in Table 10-2.

10-8 CRANK AND FOLLOWER SYNTHESIS:
FIVE ACCURACY POINTS

In the three-accuracy-point synthesis considered in Sec. 10-2, the crank and follower angles corresponding to all accuracy points were specified. If the actual angular positions of the crank and follower are left unspecified but if instead their rotations with respect to the position corresponding to the first accuracy point are given, then two additional design parameters, the crank and follower angles \( \phi_i \) and \( \psi_i \), may be considered in the synthesis. Since the number of design parameters is now five, viz., the three ratios \( a_1/a_4, a_2/a_4, a_3/a_4 \) and the two angles \( \phi_i \) and \( \psi_i \), a synthesis with five accuracy points may be expected. Let the crank and follower angles for the five accuracy points be \( \phi_i \) and \( \psi_i \) \((j = 1, 2, 3, 4, 5)\); then

\[ \phi_i = \phi_1 + \phi_{ij} \quad \text{with} \quad \phi_{11} = 0 \]

and

\[ \psi_i = \psi_1 + \psi_{ij} \quad \text{with} \quad \psi_{11} = 0 \]

where \( \phi_{ij} \) and \( \psi_{ij} \) are the rotations of the crank and follower relative to the first accuracy point.
When these values are substituted into the displacement equation of the four-bar linkage (10-3), they yield a system of equations

\[
\frac{a_4}{a_3} \cos (\phi_1 + \phi_{1\alpha}) - \frac{a_4}{a_1} \cos (\psi_1 + \psi_{1\alpha}) + \frac{a_1^2 - a_2^2 + a_3^2 + a_4^2}{2a_1a_3} = \cos (\phi_1 - \psi_1 + \phi_{1\alpha} - \psi_{1\alpha}) \quad j = 1, 2, 3, 4, 5
\]

with five unknowns,

\[
\frac{a_1}{a_1} \quad \frac{a_2}{a_4} \quad \frac{a_3}{a_4} \quad \phi_1 \quad \psi_1
\]

As in the case of the synthesis of the slider-crank mechanism with four accuracy points (Sec. 10-7), the solution of the above system does not reduce to that of linear equations; and a compatibility equation, of third degree in this case, must be considered. The solution is lengthy, but a digital-computer program giving a complete solution of the problem has been written,\(^1\) and this program may be used in most cases without an understanding of the details of the solution performed.

Consider the problem of generating the function \(y = f(x)\) in the interval \(x_s \leq x \leq x_f\) by means of a four-bar linkage \(O_AABO_B\) (Fig. 10-10), with five accuracy points such that the structural error is minimized. As usual, the variables \(x\) and \(y\) are represented by the crank and follower rotations through the relations

\[
\frac{\phi - \phi_s}{\phi_f - \phi_s} = \frac{x - x_s}{x_f - x_s} \quad \text{and} \quad \frac{\psi - \psi_s}{\psi_f - \psi_s} = \frac{y - y_s}{y_f - y_s}
\]

where \(y_s = f(x_s)\) and \(y_f = f(x_f)\). The ranges of variation \(\Delta \phi = \phi_f - \phi_s\) and \(\Delta \psi = \psi_f - \psi_s\) are chosen arbitrarily, and five accuracy points are

\(^1\) Freudenstein, second reference in the Bibliography at the end of the chapter.

**Figure 10-10** Four-bar function generator with five accuracy points, showing parameters and variables used in synthesis.
selected along the curve of \( y = f(x) \) for \( x = x_j \) (\( j = 1, 2, 3, 4, 5 \)) in the interval between \( x_s \) and \( x_f \). The crank and follower rotations from the position corresponding to the first accuracy point to the positions corresponding to the other accuracy points are then

\[
\phi_{ij} = \frac{x_j - x_1}{x_f - x_s} \Delta \phi \quad \text{and} \quad \psi_{ij} = \frac{y_j - y_1}{y_f - y_s} \Delta \psi
\]

with \( y_j = f(x_j) \) and \( j = 1, 2, 3, 4, 5 \).

A complete solution of the problem consists in solving the system of equations (10-13) to find the parameters \( a_1/a_4, a_2/a_4, a_3/a_4, \phi_1, \) and \( \psi_1 \); analyzing the linkage to determine the structural error as a function of \( x \); respacing the accuracy points in order to reduce the structural error; and solving Eq. (10-13) again. This process is repeated until the structural error is minimized. Since there are five accuracy points in the interval between \( x_s \) and \( x_f \), the structural error will have the general appearance shown in Fig. 10-11. It will be zero at each accuracy point and will reach a series of maxima and minima between the accuracy points as well as at the beginning and end of the interval. For the curve shown in Fig. 10-11, the structural error \( \epsilon s4 \), between the accuracy points \( x_3 \) and \( x_4 \), has the greatest magnitude. In respacing the accuracy points, points \( x_5 \) and \( x_4 \) must therefore be brought closer together in order to reduce \( \epsilon s4 \) at the cost of other maxima or minima. The structural error is minimized when the spacing is such that all maxima and minima are of equal magnitude.

The computer program referred to above considers the complete problem: the input data consist of the function \( y = f(x) \); the initial and final values of \( x \) to be considered, \( x_s \) and \( x_f \); the ranges of rotation of the crank and follower, \( \Delta \phi \) and \( \Delta \psi \); and the five values \( x_1, x_2, x_3, x_4, x_5 \) corresponding to the five accuracy points to be used in the first cycle of computations. With this program, the four-bar linkages shown in Table 10-3\(^1\) have been designed to generate elementary functions within speci-

\(^1\) This is table 1 from F. Freudenstein, Four-bar Function Generators, \textit{Trans. Fifth Conf. Mechanisms}, Purdue University, 1958; also \textit{Machine Design, Nov. 27, 1958}. Reprinted by courtesy of Penton Publishing Company.
<table>
<thead>
<tr>
<th>Function</th>
<th>( \log z )</th>
<th>( \sin z )</th>
<th>( \tan z )</th>
<th>( e^z )</th>
<th>( \frac{1}{z} )</th>
<th>( z^{1.5} )</th>
<th>( z^2 )</th>
<th>( z^{1.5} )</th>
<th>( z^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interval of ( z )</td>
<td>( 1 \leq z \leq 2 )</td>
<td>( 0 \leq z \leq 90 \text{ deg} )</td>
<td>( 0 \leq z \leq 45 \text{ deg} )</td>
<td>( 0 \leq z \leq 1 )</td>
<td>( 1 \leq z \leq 2 )</td>
<td>( 0 \leq z \leq 1 )</td>
<td>( 0 \leq z \leq 1 )</td>
<td>( 0 \leq z \leq 1 )</td>
<td>( -1 \leq z \leq +1 )</td>
</tr>
<tr>
<td>Range of ( \phi ), deg</td>
<td>60</td>
<td>90</td>
<td>90</td>
<td>90</td>
<td>90</td>
<td>90</td>
<td>90</td>
<td>90</td>
<td>60</td>
</tr>
<tr>
<td>Range of ( \psi ), deg</td>
<td>60</td>
<td>90</td>
<td>90</td>
<td>90</td>
<td>90</td>
<td>90</td>
<td>90</td>
<td>90</td>
<td>60</td>
</tr>
<tr>
<td>Accuracy Points:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x_1 )</td>
<td>1.0144678</td>
<td>2.7633673</td>
<td>1.0246052</td>
<td>0.0278109</td>
<td>1.0130685</td>
<td>0.111835966</td>
<td>0.033687272</td>
<td>0.022320458</td>
<td>0.029325922</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>1.1555598</td>
<td>21.985025</td>
<td>12.42451</td>
<td>0.15327052</td>
<td>1.156103</td>
<td>0.14032685</td>
<td>0.24917564</td>
<td>0.21794875</td>
<td>0.25920473</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>1.3843747</td>
<td>48.228992</td>
<td>27.476764</td>
<td>0.47670748</td>
<td>1.4981776</td>
<td>0.40950302</td>
<td>0.54280174</td>
<td>0.54157659</td>
<td>0.58821332</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>1.7049555</td>
<td>71.414168</td>
<td>38.542302</td>
<td>0.79659399</td>
<td>1.7984797</td>
<td>0.74496432</td>
<td>0.8186346</td>
<td>0.8186346</td>
<td>0.84128244</td>
</tr>
<tr>
<td>( x_5 )</td>
<td>1.9627009</td>
<td>87.54952</td>
<td>44.249088</td>
<td>0.97582616</td>
<td>1.9786677</td>
<td>0.97029835</td>
<td>0.97883119</td>
<td>0.97705758</td>
<td>0.98039595</td>
</tr>
<tr>
<td>Crank Angle ( \phi_1 ), deg and decimals</td>
<td>52.628390</td>
<td>-62.283261</td>
<td>269.70917</td>
<td>241.64463</td>
<td>33.804400</td>
<td>-5.170711</td>
<td>-29.320694</td>
<td>-88.314667</td>
<td>-85.921508</td>
</tr>
<tr>
<td>Follower Angle ( \psi_1 ), deg and decimals</td>
<td>259.07749</td>
<td>75.606709</td>
<td>124.18966</td>
<td>40.422765</td>
<td>120.21283</td>
<td>211.68878</td>
<td>233.83630</td>
<td>44.492538</td>
<td>37.636724</td>
</tr>
<tr>
<td>Link Proportions:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( a_6/a_4 )</td>
<td>-3.23458</td>
<td>1.8343688</td>
<td>-2.560319</td>
<td>-3.4994589</td>
<td>-0.35477780</td>
<td>0.62481545</td>
<td>2.5233585</td>
<td>-1.800087</td>
<td>-1.6056207</td>
</tr>
<tr>
<td>( a_6/a_4 )</td>
<td>0.84550215</td>
<td>2.2835439</td>
<td>7.4304444</td>
<td>0.87854083</td>
<td>1.0305418</td>
<td>1.3088026</td>
<td>3.3295928</td>
<td>0.90892980</td>
<td>0.92537009</td>
</tr>
<tr>
<td>( a_6/a_4 )</td>
<td>3.48551187</td>
<td>-0.69364556</td>
<td>8.6856194</td>
<td>3.3996445</td>
<td>0.38453884</td>
<td>-0.40060612</td>
<td>-0.5554832</td>
<td>1.2737203</td>
<td>1.1073831</td>
</tr>
<tr>
<td>Angular Error, deg and decimals</td>
<td>0.0037</td>
<td>0.1900</td>
<td>0.038</td>
<td>0.0258</td>
<td>0.0161</td>
<td>0.146</td>
<td>0.0673</td>
<td>0.412</td>
<td>0.5035</td>
</tr>
<tr>
<td>Output Error, % Output Travel</td>
<td>0.0062</td>
<td>0.21</td>
<td>0.042</td>
<td>0.0287</td>
<td>0.0179</td>
<td>0.162</td>
<td>0.0748</td>
<td>0.457</td>
<td>0.556</td>
</tr>
</tbody>
</table>
fied intervals. The table shows the accuracy points corresponding to optimum spacing, i.e., for minimum structural error as well as the magnitude of this error. Note that the error varies widely from function to function and depends on the interval of generation as well. For example, function \( y = x^2 \) in the interval \( 0 \leq x \leq 1 \) is generated with a structural error less than 0.075 percent of the output travel. When the interval is extended to \( -1 \leq x \leq +1 \), the error becomes 4.47 percent, or 60 times as large. The large structural error in this case is due to the symmetry of the function \( y = x^2 \) in the interval \( -1 \leq x \leq +1 \) (Fig. 10-12). The four-bar linkage is not suited to the generation of symmetrical functions. Mechanisms better suited to this purpose will be considered in Chap. 12.

10-9 ANALYSIS OF MECHANICAL ERRORS IN LINKAGES

Consider a linkage with \( n \) constant parameters \( q_1, q_2, \ldots, q_n \), transforming a motion (as shaft rotation), defined by an input variable \( \phi \), into another motion, defined by an output variable \( \psi \). This linkage has been designed to generate a given function in a given interval such that, if the linkage were built to perfection, the maximum deviation between the desired function and the function generated by the linkage would not exceed \( \varepsilon_s \), the mathematical or structural error. This deviation is present in all linkages designed by approximate synthesis. An additional error, \( \varepsilon_m \), due to deflections of the links, play in the joints, and manufacturing tolerances, will inevitably occur in addition to \( \varepsilon_s \) in any actual linkage. This mechanical error \( \varepsilon_m \) will now be evaluated in terms of dimensional variations that may be maintained on the values of the parameters \( q_1, q_2, \ldots, q_n \).

The displacement equation of the linkage, relating the constant parameters \( q_1, q_2, \ldots, q_n \) to the input and output variables \( \phi \) and \( \psi \), may be written in general form as

\[
F(q_1, q_2, \ldots, q_n, \phi, \psi) = 0
\] (10-14)
The errors of the constant parameters \( q_1, q_2, \ldots, q_n \) are assumed to be \( \Delta q_1, \Delta q_2, \ldots, \Delta q_n \). For a given value of the input variable \( \phi \), the value of the output variable will be \( \psi + \Delta \psi \), and we may write

\[
F(q_1 + \Delta q_1, q_2 + \Delta q_2, \ldots, q_n + \Delta q_n, \phi, \psi + \Delta \psi) = 0 \quad (10-15)
\]

The mechanical error manifested at the output is

\[
\epsilon_m = \Delta \psi
\]

For small values of the errors \( \Delta q_1, \Delta q_2, \ldots, \Delta q_n \) and \( \Delta \psi \), the differential of the function \( F \) may be written in terms of its partial derivatives as

\[
\frac{\partial F}{\partial q_1} \Delta q_1 + \frac{\partial F}{\partial q_2} \Delta q_2 + \cdots + \frac{\partial F}{\partial q_n} \Delta q_n + \frac{\partial F}{\partial \psi} \Delta \psi = 0
\]

or

\[
\epsilon_m = - \sum_{k=1}^{n} \frac{\frac{\partial F}{\partial q_k}}{\frac{\partial F}{\partial \psi}} \Delta q_k
\]

The total mechanical error \( \epsilon_m \) in the linkage is therefore the sum of the individual errors due to each of the parameters considered separately.

10-10 MECHANICAL ERRORS IN FOUR-BAR LINKAGES

For our present purpose, the displacement equation (10-1) of the four-bar linkage is written as

\[
D \sin \psi + E \cos \psi = F
\]

in which

\[
D = 2a_1a_3 \sin \phi \\
E = 2a_3a_4 + 2a_1a_3 \cos \phi \\
F = 2a_1a_4 \cos \phi + a_1^2 - a_2^2 + a_3^2 + a_4^2
\]

Errors \( \Delta a \) in the link lengths \( a_1, a_2, a_3, a_4 \) will modify the coefficients \( D \) and \( E \) and the term \( F \) by amounts \( \Delta D, \Delta E, \) and \( \Delta F \) and will produce an error \( \Delta \psi \) in the output. Each error \( \Delta a \) in a given link will produce separate errors in \( D, E, \) and \( F, \) whence the output error contributed by each link must be considered separately. The total mechanical error \( \epsilon_m \) of the linkage will be the sum of the separate errors.
Link-error Equation

In the presence of link-length errors the displacement equation (10-16) may be written as

\[(D + \Delta D) \sin (\psi + \Delta \psi) + (E + \Delta E) \cos (\psi + \Delta \psi) = F + \Delta F\]  

(10-17)

After expansion of this equation (by use of trigonometric identities, small-angle approximations, and neglect of higher-order terms), the subtraction of Eq. (10-16), and ordering of terms, we find

\[\Delta \psi (D \cos \psi - E \sin \psi) = -\Delta D \sin \psi - \Delta E \cos \psi + \Delta F\]

from which \[\Delta \psi = -\frac{\Delta D \sin \psi + \Delta E \cos \psi - \Delta F}{D \cos \psi - E \sin \psi} = (\epsilon_m)_{\text{link}}\]  

(10-18)

This is the link-error equation.

Error Due Only to $\Delta a_1$

An error $\Delta a_1$ in link dimension $a_1$ produces deviations $\Delta D$, $\Delta E$, and $\Delta F$ in $D$, $E$, and $F$ to yield a mechanical error $\epsilon_m = \Delta \psi_1$. The deviation $\Delta D$ is found as follows:

\[\Delta D = (D + \Delta D) - D = [2(a_1 + \Delta a_1)a_3 \sin \phi] - 2a_1a_3 \sin \phi\]
\[= (2a_3 \sin \phi) \Delta a_1\]

In like manner

\[\Delta E = (2a_3 \cos \phi) \Delta a_1\]
\[\Delta F = (2a_4 \cos \phi + 2a_1) \Delta a_1\]

Substitution of these values into Eq. (10-18) yields

\[\Delta \psi_1 = \epsilon_m = 2 \frac{a_1 \cos \phi + a_1 - a_3 \cos (\phi - \psi)}{D \cos \psi - E \sin \psi} \Delta a_1\]

Error Due Only to $\Delta a_2$

The deviations are

\[\Delta D = 0 \quad \Delta E = 0 \quad \Delta F = -(2a_2) \Delta a_2\]

and

\[\Delta \psi_2 = \epsilon_m = \frac{-2a_2}{D \cos \psi - E \sin \psi} \Delta a_2\]
Error Due Only to $\Delta a_3$

Here

$$\Delta D = (2a_1 \sin \phi) \Delta a_3 \quad \Delta E = (2a_4 + 2a_1 \cos \phi) \Delta a_3 \quad \Delta F = 2a_3 \Delta a_3$$

and

$$\Delta \psi_3 = \epsilon_{m3} = -\left(2 \frac{a_4 \cos \phi + a_3 - a_1 \cos \psi}{D \cos \psi - E \sin \psi}\right) \Delta a_3$$

Error Due Only to $\Delta a_4$

Here

$$\Delta D = 0 \quad \Delta E = 2a_3 \Delta a_4 \quad \Delta F = (2a_1 \cos \phi + 2a_4) \Delta a_4$$

and

$$\Delta \psi_4 = \epsilon_{m4} = 2 \frac{a_1 \cos \phi + a_4 - a_3 \cos \psi}{D \cos \psi - E \sin \psi} \Delta a_4$$

Numerical Example  Consider the four-bar linkage designed by the five-point synthesis method to generate the function log $x$ for values of $x$ between 1 and 2. The dimensions of this linkage, given in Table 10-3, are approximately

$$a_1 = -3.23 \text{ in.} \quad a_2 = 0.84 \text{ in.} \quad a_3 = 3.48 \text{ in.} \quad a_4 = 1.0 \text{ in.}$$

with the first accuracy point at

$$\phi_1 = 53^\circ \quad \psi_1 = 259^\circ$$

and a maximum structural error $\epsilon_s = 0.0037^\circ$. The mechanical error will now be evaluated at the first accuracy point in terms of errors $\Delta a_1$, $\Delta a_2$, $\Delta a_3$, $\Delta a_4$ that may be assumed for the link lengths $a_1, a_2, a_3, a_4$. At the first accuracy point,

$$D = -18.7 \quad E = -7.04 \quad F = 19.64$$

$$\cos \psi = -0.19 \quad \sin \psi = -0.98$$

from which

$$D \cos \psi - E \sin \psi = -3.38$$

$$\epsilon_{m1} = -0.24 \Delta a_1 \text{ rad} = -14 \Delta a_1 \text{ deg}$$

$$\epsilon_{m2} = 0.50 \Delta a_2 \text{ rad} = 29 \Delta a_2 \text{ deg}$$

$$\epsilon_{m3} = -0.38 \Delta a_3 \text{ rad} = -23 \Delta a_3 \text{ deg}$$

$$\epsilon_{m4} = 0.20 \Delta a_4 \text{ rad} = 12 \Delta a_4 \text{ deg}$$

Assuming $|\Delta a_1| = |\Delta a_2| = |\Delta a_3| = |\Delta a_4| = 0.001 \text{ in.},$

$$|\epsilon_m|_{\text{max}} = |\epsilon_{m1}| + |\epsilon_{m2}| + |\epsilon_{m3}| + |\epsilon_{m4}| = 0.001(78) = 0.078^\circ$$

or

$$|\epsilon_m|_{\text{rms}} = \sqrt{\epsilon_{m1}^2 + \epsilon_{m2}^2 + \epsilon_{m3}^2 + \epsilon_{m4}^2} = 0.001(42) = 0.042^\circ$$

The maximum mechanical error with the 0.001-in. tolerance is thus more than 20 times the structural error (0.0780/0.0038 $\simeq 20$); the ratio of the rms error is more than 11 times the structural (0.0420/0.0037 $\simeq 11$).
10-11 GEOMETRIC INTERPRETATION OF THE ERROR DENOMINATOR

Intuition and experience indicate that the errors in a four-bar linkage are closely related to the value assumed by the angle $\gamma$ (Fig. 10-13), already identified in Sec. 2-10 as the transmission angle. The role played by the angle $\gamma$ in the mechanical errors and force transmission of the mechanism indicates that there should be a relation between this angle and the denominator, $G = D \cos \psi - E \sin \psi$, in the expressions of the mechanical errors.

The angle $\gamma$ may be expressed in terms of the linkage parameters and the input variable $\phi$ by application of the cosine law to triangles $O_AO_BA$ and $ABO_B$ (Fig. 10-13),

\[(AO_B)^2 = a_2^2 + a_3^2 - 2a_2a_3 \cos \gamma = a_1^2 + a_4^2 + 2a_1a_4 \cos \phi\]

or \[\cos \gamma = -\frac{a_1^2 - a_2^2 - a_3^2 + a_4^2 + 2a_1a_4 \cos \phi}{2a_2a_3}\] (10-19)

To express $G$ in terms of the same quantities, the angle $\psi$ must be eliminated by using the displacement equation of the linkage [Eq. (10-1)]. Solving simultaneously the equations

\[
\begin{align*}
D \sin \psi + E \cos \psi &= F \\
-E \sin \psi + D \cos \psi &= G
\end{align*}
\]

yields

\[
\begin{vmatrix}
F & E \\
G & D
\end{vmatrix} \quad \begin{vmatrix}
D & F \\
-E & G
\end{vmatrix}
\]

\[
\begin{vmatrix}
D & E \\
-E & D
\end{vmatrix}
\]

\[
\sin \psi = \frac{D}{\sqrt{D^2 + E^2}} \quad \cos \psi = \frac{E}{\sqrt{D^2 + E^2}}
\]

**Figure 10-13** Determination of transmission angle $\gamma$. 

...
which by substitution into the trigonometric identity
\[ \sin^2 \psi + \cos^2 \psi = 1 \]
gives
\[ \begin{vmatrix} F & E \\ G & D \end{vmatrix}^2 + \begin{vmatrix} D & F \\ -E & G \end{vmatrix}^2 = \begin{vmatrix} D & E \\ -E & D \end{vmatrix}^2 \]
Expanding the determinants, collecting terms, and dividing through by \( D^2 + E^2 \) yields
\[ G^2 = D^2 + E^2 - F^2 \] (10-20)
Note that, according to this relation, \( G \) is the square root which appears in the expressions of \( \psi^+ \) and \( \psi^- \) derived in Sec. 10-1. When \( G = 0 \), \( \psi^+ = \psi^- \) and the linkage is in dead-center position, whence no torque can be transmitted from the crank to the follower.

Introducing the expressions for \( D, E, \) and \( F \) (Sec. 10-10) into Eq. (10-20) yields
\[
G^2 = 4a_1^2a_3^2 \sin^2 \phi + (2a_3a_4 + 2a_1a_2 \cos \phi)^2
- (2a_1a_4 \cos \phi + a_1^2 - a_2^2 + a_3^2 + a_4^2)^2
\]
By combination of terms and algebraic simplifications, this reduces to
\[
G^2 = -4a_2^2a_3^2 \frac{2a_1a_4 \cos \phi + a_1^2 - a_2^2 - a_3^2 + a_4^2}{2a_2a_3} + 4a_2^2a_3^2
\]
or
\[
G^2 = 4a_2^2a_3^2(-\cos^2 \gamma + 1)
\]
or
\[
G = \pm 2a_2a_3 \sin \gamma \] (10-21)
This equation shows that the quantity \( G \) and the angle \( \gamma \) are indeed two equivalent expressions of the same property of a four-bar linkage. It confirms the fact that a four-bar linkage having poor force transmission is also subject to large mechanical errors.

**BIBLIOGRAPHY**
