8 GEOMETRIC METHODS OF SYNTHESIS WITH THREE ACCURACY POINTS

8-1 INTRODUCTION

The geometric methods for the synthesis of planar linkages for various duties antedate the more precise algebraic attacks. They still occupy a most important place among the available procedures, for they are relatively fast in producing answers and, since they maintain touch with physical reality to a much greater degree than do the algebraic methods, are often more readily understood. Also, their degree of accuracy is adequate for many situations and they are valuable adjuncts to the algebraic methods, for their approximate solutions serve as useful guides in directing the course of equations or in reaching certain decisions.

The relatively long development period of the geometric methods has led to many techniques hand-tailored to particular types of problems. It is the intent of this chapter and the next to dwell on the general and avoid the particular, i.e., to present general concepts that may be applied to a wide variety of problems and abstain from peculiarly specialized techniques.

The synthesis of linkages involves moving a link from a first position to several others, sometimes under specifications of velocity and acceleration. A sequence of problems will be used to display basic geometric concepts. The solutions of the first and simpler problems of this chapter will establish relations to be considered further in the next and will be applied to more complex situations.

8-2 POLES OF THE FOUR-BAR LINKAGE

Problem 1 Arrange for the transfer of a link AB from position A_1B_1 to a second position A_2B_2.

This problem may be solved in different ways, as shown in Fig. 8-1. For example, the midnormals a_{12} and b_{12} to the distances A_1A_2 and B_1B_2 will intersect at P_{12}. Links $P_{12}A_1$ and $P_{12}B_1$, connected as shown in Fig. 8-1*a*, will allow AB to assume its two positions. This solution—one may think of $A_1P_{12}B_1$ as a solid triangle pivoted about P_{12}—is trivial but recalls that all lines associated with the plane of link AB undergo the same rotation; i.e., the angles through which they turn are equal in magnitude and sense of rotation. The half angles $A_1P_{12}a_{12}$ and $B_1P_{12}b_{12}$ are also equal.

The transfer of the link AB may also be carried out by a four-bar linkage with AB as coupler. The centers of the fixed revolutes O_A and O_B may be chosen anywhere along a_{12} and b_{12}, respectively, the other two revolute centers being located at A and B on the moving link (Fig. 8-1*b*). Note that this solution to the problem involves two independent choices with revolute centers at A and B on the moving link. For each center O_A chosen on a_{12} there are an infinity of solutions corresponding to different choices of O_B along b_{12}. But since there are also an infinity of choices possible for O_A along a_{12}, it may be said that the present problem has an infinity to the square number of solutions, denoted as ∞^2.

If either O_A or O_B is chosen at infinity along its midnormal (Fig. 8-1*c* and *d*), the corresponding four-bar linkages change into slider-crank mechanisms. The number of solutions for each situation is infinite because of the infinite choices that are available for locating the fixed revolute (O_A or O_B, as the case may be).

If both O_A and O_B are chosen at infinity along their midnormals (Fig. 8-1*e*), a *PPRR* mechanism results and the solution is unique.

In the foregoing a link AB, that is, a portion of a plane with elements (connections) at A and at B, was caused to move from one position to another. If the specification is altered to moving a plane containing the *line* AB from one position to another, the number of solutions for each of the previous situations is multiplied by ∞^4. This addition comes from having to choose two points C and D of the plane as locations for the

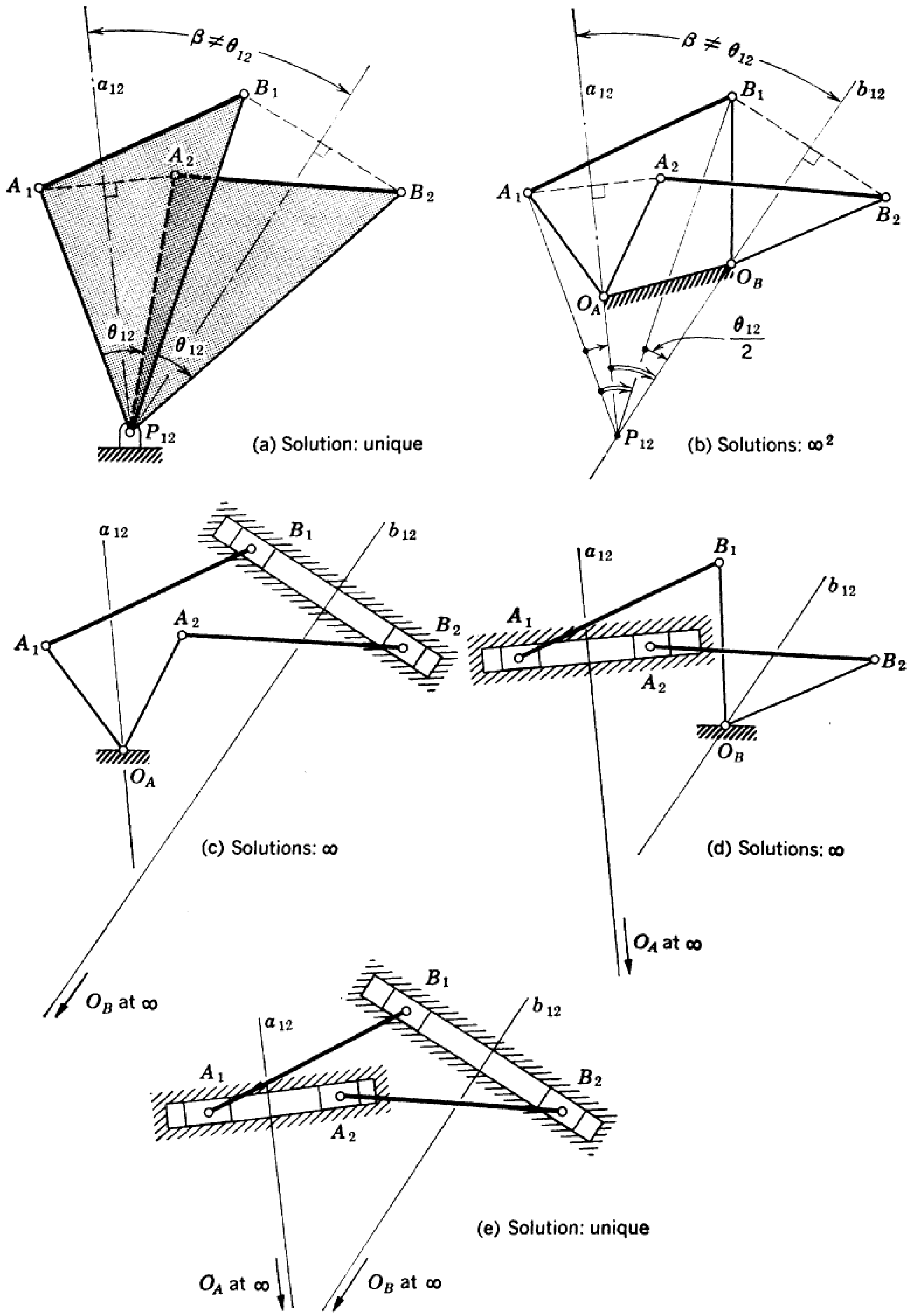

FIGURE 8-1 Problem 1, two positions of a link.

revolute elements; each point has two coordinates, whence four choices must be made.

DEFINITION The point P_{12}, the center for the finite rotation of AB from A_1B_1 to A_2B_2, is the *pole* of the rotation θ_{12}.

The pole P_{12} is a property of the finite displacement of AB from A_1B_1 to A_2B_2 and is independent of how the link actually moves between these two positions. However, if A_2B_2 is made to approach A_1B_1, so that A_1B_1 and A_2B_2 become two infinitesimally close positions, the pole P_{12} then becomes the familiar *instantaneous center* of rotation of link AB at the instant considered. Figure 8-1*b* depicts potentially useful angular relations that should be noted, namely, $A_1P_{12}A_2 = B_1P_{12}B_2 = \theta_{12}$ and their half angles $A_1P_{12}a_{12} = B_1P_{12}b_{12} = \theta_{12}/2$. This observation leads to the following theorem for four-bar linkages:

THEOREM I When viewed from a pole of rotation, the coupler and frame are seen under angles that either are equal or differ from each other by 180°; this is true for the two positions defining the pole. Similarly, the crank and follower are seen from the pole under angles that either are equal or differ from each other by 180° for the two positions.

A situation in which the angles are equal is shown in Fig. 8-1*b*; angles differing by 180° are shown in Fig. 8-2.

Problem 2 Design a four-bar linkage to transfer a link AB through three specified positions A_1B_1, A_2B_2, A_3B_3 (Fig. 8-3).

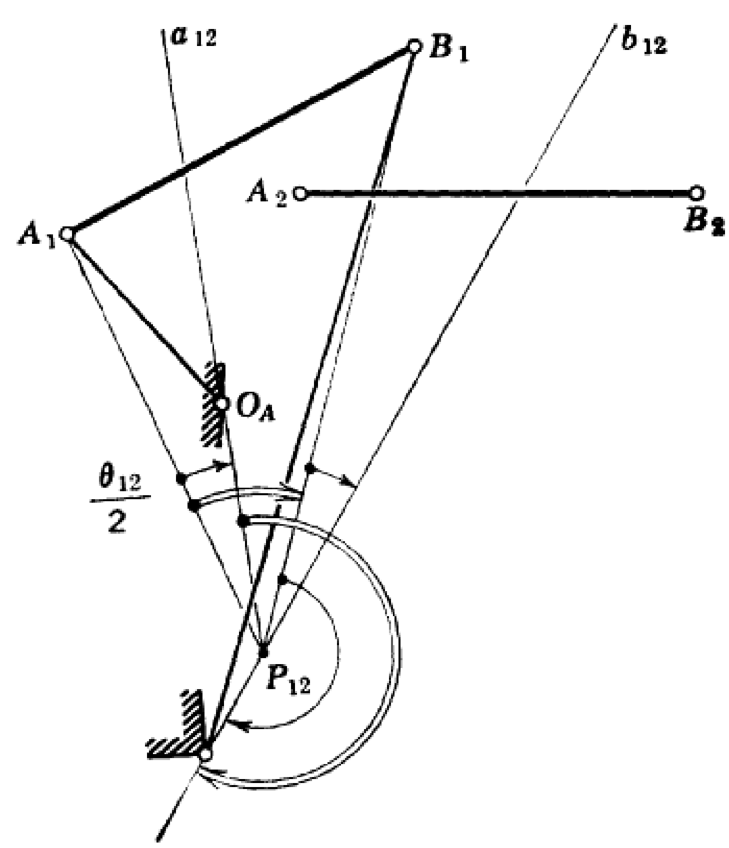

FIGURE 8-2 To Theorem I.

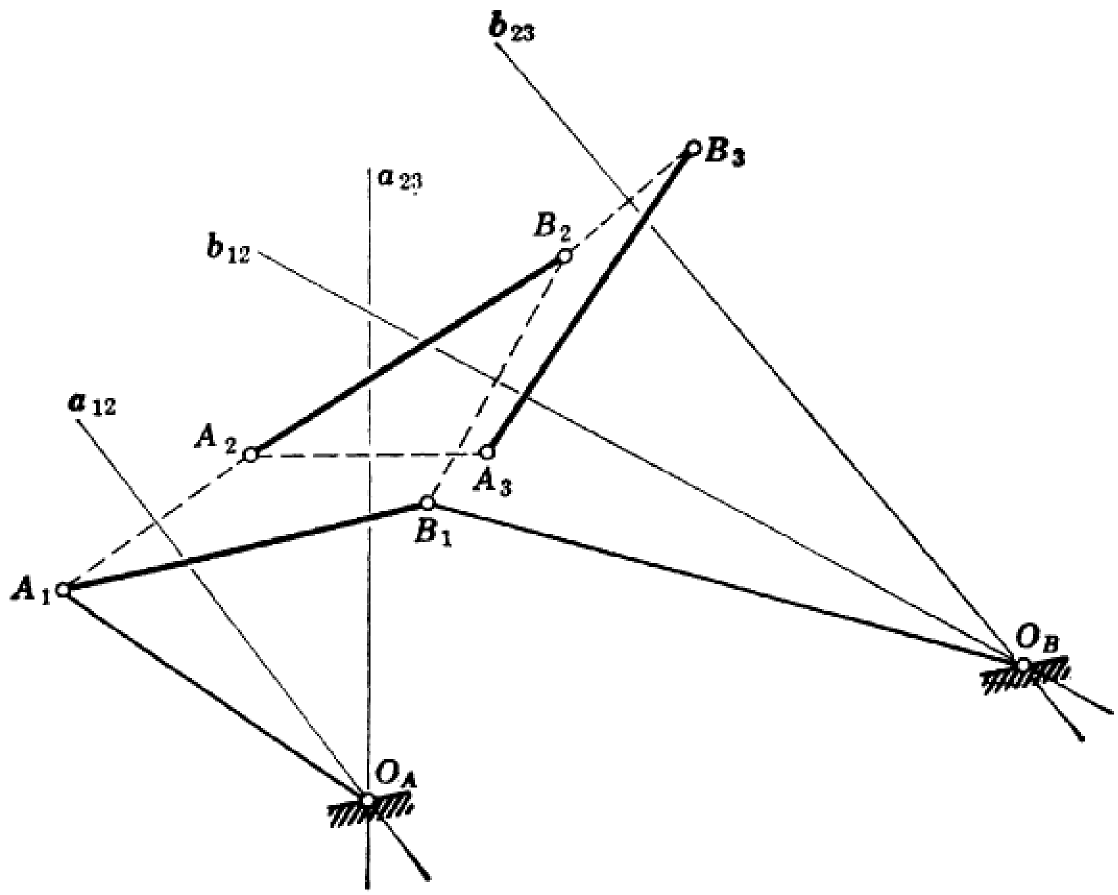

FIGURE 8-3 Problem 2, three positions of a link.

The centers of the fixed revolutes O_A and O_B are uniquely defined here as the intersections of a_{12}, a_{23} and b_{12}, b_{23}, respectively (Fig. 8-3). If A and B are the revolute centers of the coupler, this problem has a unique solution.

Problem 3 Design a four-bar linkage of frame O_AO_B, in which a given position ϕ and a given angular velocity ω_2 of the crank produce a specified position ψ and a specified angular velocity ω_4 of the follower.

The centers of the fixed revolutes being O_A and O_B (Fig. 8-4), the specified positions of the crank and follower center lines are defined by angles ϕ and ψ, and the revolute centers of the coupler must lie on these centerlines. The velocity of A is perpendicular to O_AA with a magnitude

$$v_A = (O_AA)\omega_2 \tag{8-1}$$

Similarly, the velocity of B is perpendicular to O_BB with a magnitude

$$v_B = (O_BB)\omega_4 \tag{8-2}$$

and the instantaneous center of rotation of the couple AB, at this instant, is the intersection I of O_AA and O_BB. But since I is the instantaneous center of AB, AB rotates at this instant about I with an angular velocity ω_3; thus

$$v_A = (IA)\omega_3 \qquad \text{and} \qquad v_B = (IB)\omega_3$$

or

$$\frac{v_A}{v_B} = \frac{IA}{IB}$$

Substituting the values of v_A and v_B from Eqs. (8-1) and (8-2) into this last equation yields

$$\frac{(O_AA)\omega_2}{(O_BB)\omega_4} = \frac{IA}{IB}$$

or

$$\frac{IB}{O_BB} = \frac{IA}{O_AA}\frac{\omega_4}{\omega_2} \tag{8-3}$$

The solution may be summarized as follows:

1. Lay out the frame and construct the instantaneous center I corresponding to ϕ and ψ.
2. Choose A arbitrarily along O_AI.
3. Evaluate the ratio IB/O_BB from Eq. (8-3).
4. Construct point B along O_BI so that it divides the segment O_BI in the above ratio.

8-3 RELATIVE POLES OF THE FOUR-BAR LINKAGE

In moving a specified coupler from one position to another, we considered the coupler from the vantage of a point called the pole P_{12}, determined from the midnormals of the two coupler positions (Fig. 8-1*b*). This pole, common to frame and coupler, gave no relation between the swing angles (or angular displacements) of the crank and follower. The crank (input) and follower displacements ϕ_{12} and ψ_{12} for a given coupler displacement are identified in Fig. 8-5*a*. To correlate ϕ_{12} and ψ_{12} through the coupler motion, we shall consider the follower motion with respect to the crank. We do this by means of a kinematic inversion: we shall

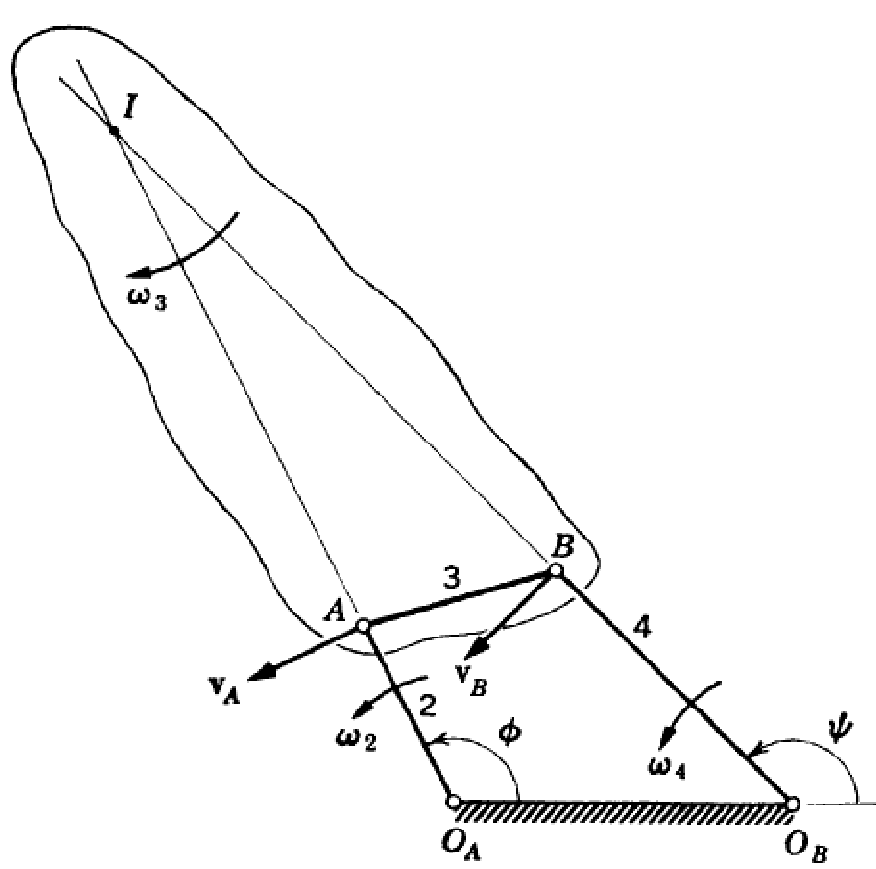

FIGURE 8-4 Problem 3, specification of angular velocities and positions of links 2 and 4.

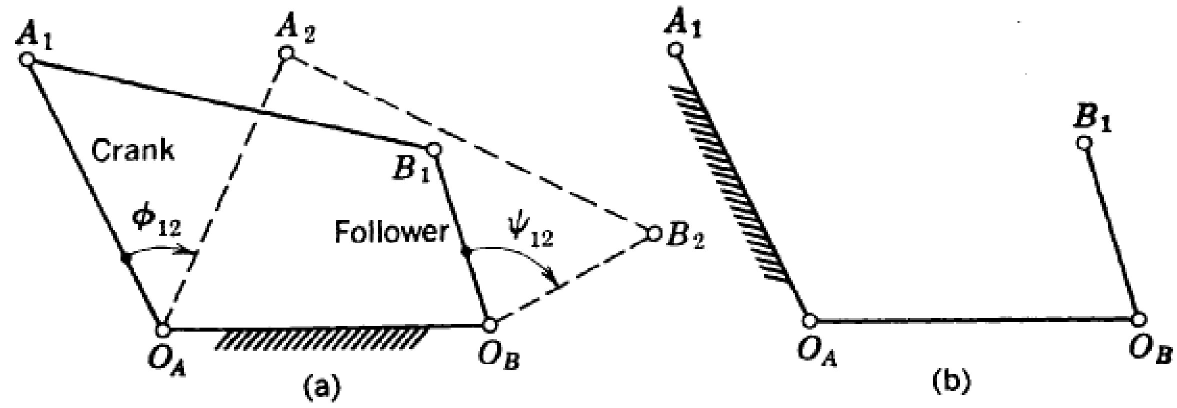

FIGURE 8-5 Four-bar linkage displacements needing correlation.

assume the crank fixed, thus becoming observers on it, and shall remember that in any displacement of a linkage the relative motions of all links remain the same, regardless of which link is fixed.

The linkage with the crank O_AA_1 fixed, in a position ready for displacement, is shown in Fig. 8-5*b*. On a linkage displacement consistent with that of Fig. 8-5*a*, the follower displacement with respect to the crank (Fig. 8-6) is seen to be the result of two separate rotations $-\phi_{12}$ and ψ_{12}. It is our purpose to combine these two follower rotations into a single equivalent rotation and find the unique point, called *relative pole*[1] R_{12}, about which the single rotation takes place.

The displacement of the follower from O_BB_1 to $O'_BB'_2$ is the result of two rotations (Fig. 8-6):

1. A rotation about O_A of angle $-\phi_{12}$, from O_BB_1 to $O'_BB'_1$. Note the negative sign: if O_AA_1 rotates clockwise with respect to O_AO_B, then O_AO_B rotates counterclockwise with respect to O_AA_1.

2. A rotation around O_B of angle ψ_{12} from $O'_BB'_1$ to $O'_BB'_2$.

The angle of rotation from O_BB_1 to $O'_BB'_2$ is thus $\psi_{12} - \phi_{12}$. The relative pole R_{12} is the intersection of the midnormals b'_{12} of $B_1B'_2$ and c'_{12}

[1] A second relative pole exists for a displacement of the crank with respect to the follower. We shall not consider it here, for we shall not need it. Its construction is similar to that of the pole which we are discussing.

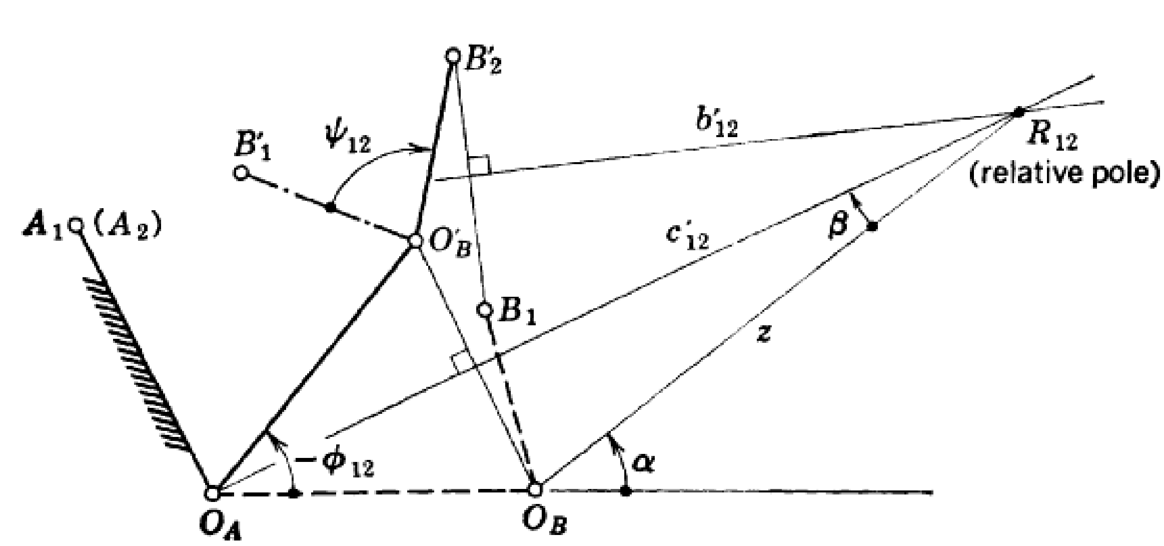

FIGURE 8-6 Definition of the relative pole.

of $O_BO'_B$, and the triangle $O_AO_BR_{12}$ yields $\alpha + \beta = -\phi_{12}/2$. Since $\beta = (\psi_{12} - \phi_{12})/2$, the half angle of rotation from O_BB_1 to $O'_BB'_2$, we may write $\alpha = -\phi_{12}/2 - (\psi_{12} - \phi_{12})/2 = -\psi_{12}/2$ and R_{12} is the intersection of the lines c'_{12} and z, making, respectively, the angles $-\phi_{12}/2$ and $-\psi_{12}/2$ with O_AO_B.

Theorem I also applies to the inverted mechanism and its relative pole R_{12}: viewed from R_{12}, O_AA_1 and O_BB_1 appear under equal angles, and O_AO_B and A_1B_1 are also seen under equal angles.

Construction of the Relative Pole for Specified ϕ_{12} and ψ_{12}

Select convenient frame points O_A and O_B, and draw two lines making the angles $-\phi_{12}/2$ and $-\psi_{12}/2$, respectively, with O_AO_B. Their intersection is R_{12}. Note again the negative signs: if ϕ_{12} is clockwise, then $-\phi_{12}/2$ is counterclockwise from O_AO_B. Similarly, if ψ_{12} is clockwise, then $-\psi_{12}/2$ is counterclockwise.

Problem 4 Design a four-bar linkage in which a given angular displacement ϕ_{12} of the crank produces a given angular displacement ψ_{12} of the follower (Fig. 8-7).

Solution

1. Assume a convenient frame O_AO_B, and construct the relative pole R_{12} corresponding to ϕ_{12} and ψ_{12}.
2. Choose point A_1 arbitrarily.
3. Draw a line $R_{12}u$ such that

$$\beta = A_1R_{12}u = O_AR_{12}O_B \qquad \text{in magnitude and direction}$$

4. Choose any point B_1 on $R_{12}u$.

By Theorem I, the desired linkage is $O_AA_1B_1O_B$. The arbitrary choices—A_1 and B_1—give the problem ∞^3 solutions. The method fails if $\phi_{12} = 0$, or $\psi_{12} = 0$, or $\phi_{12} = \psi_{12}$.

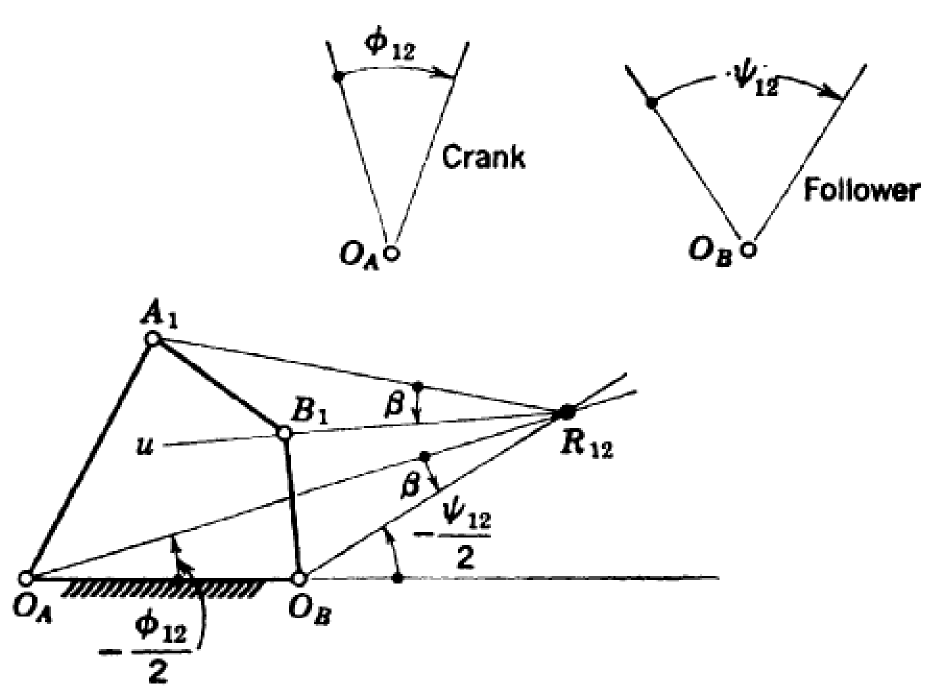

FIGURE 8-7 Solution of Prob. 4.

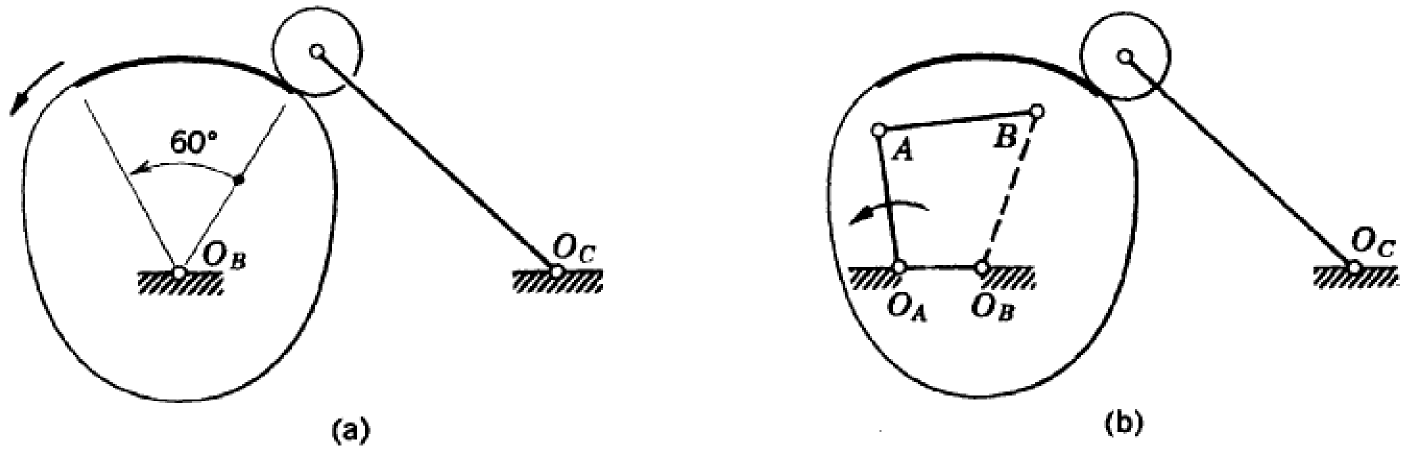

FIGURE 8-8 Modification of timing of cam mechanism.

The foregoing method may be applied to modifying the follower dwell of an existing cam mechanism without changing cams. Assume that the cam follower (Fig. 8-8*a*) has a dwell for 60° of camshaft rotation (input). It is desired to have the follower at dwell for 90° of input rotation. The solution involves slowing the present cam by making it the output link O_BB of a four-bar linkage (Fig. 8-8*b*), proportioning the linkage such that the 60° of constant cam radius corresponds to 90° of input rotation, now applied to the crank O_AA. The cam link O_BB must be able to rotate continuously; this condition demands that the four-bar be a drag-link (double-crank) configuration. Accordingly the Grashof rule $l + s < p + q$ must be observed, with the frame O_AO_B serving as the shortest link.

The determination of this linkage is shown in Fig. 8-9. After constructing the relative pole R_{12}, point A_1 is chosen so that $O_AA_1 > O_AO_B$; $R_{12}u$ is constructed as in step 3, Prob. 4, and B_1 is chosen along $R_{12}u$ such that

$$O_AA_1 + O_AO_B < A_1B_1 + O_BB_1$$

and

$$A_1B_1 < O_AA_1$$

The desired linkage is $O_AA_1B_1O_B$, also shown in Fig. 8-9.

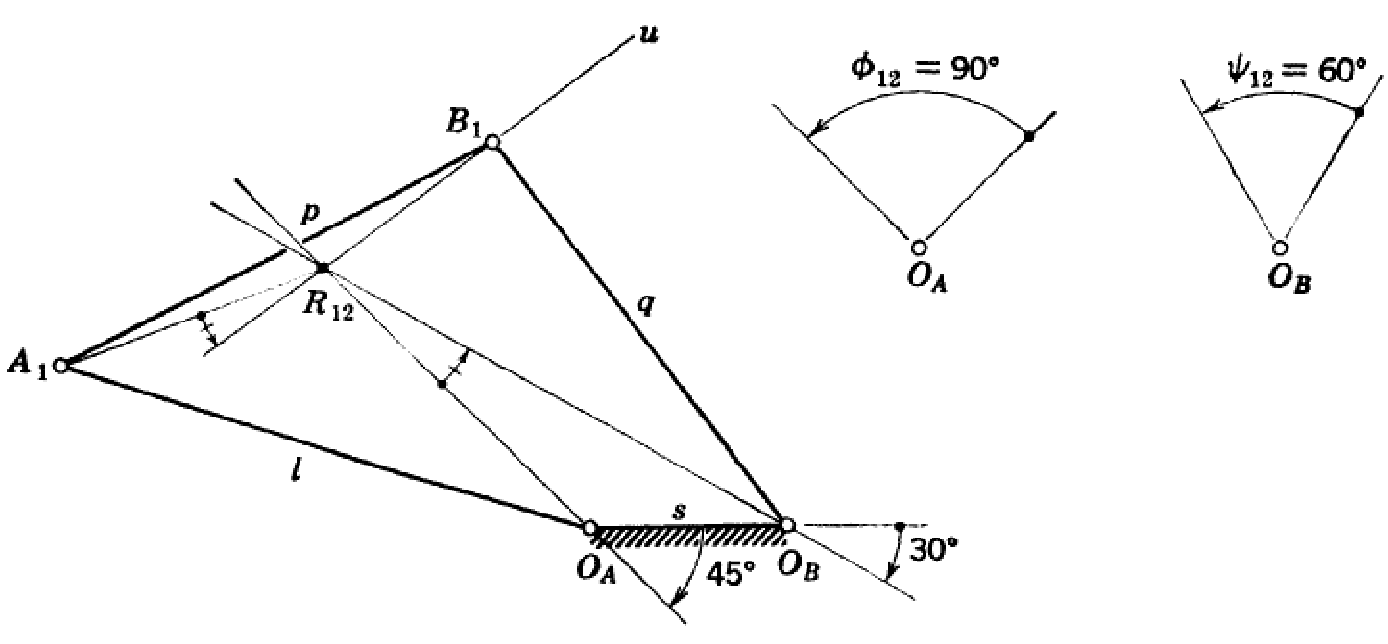

FIGURE 8-9 Modification of timing of cam mechanism; construction of linkage.

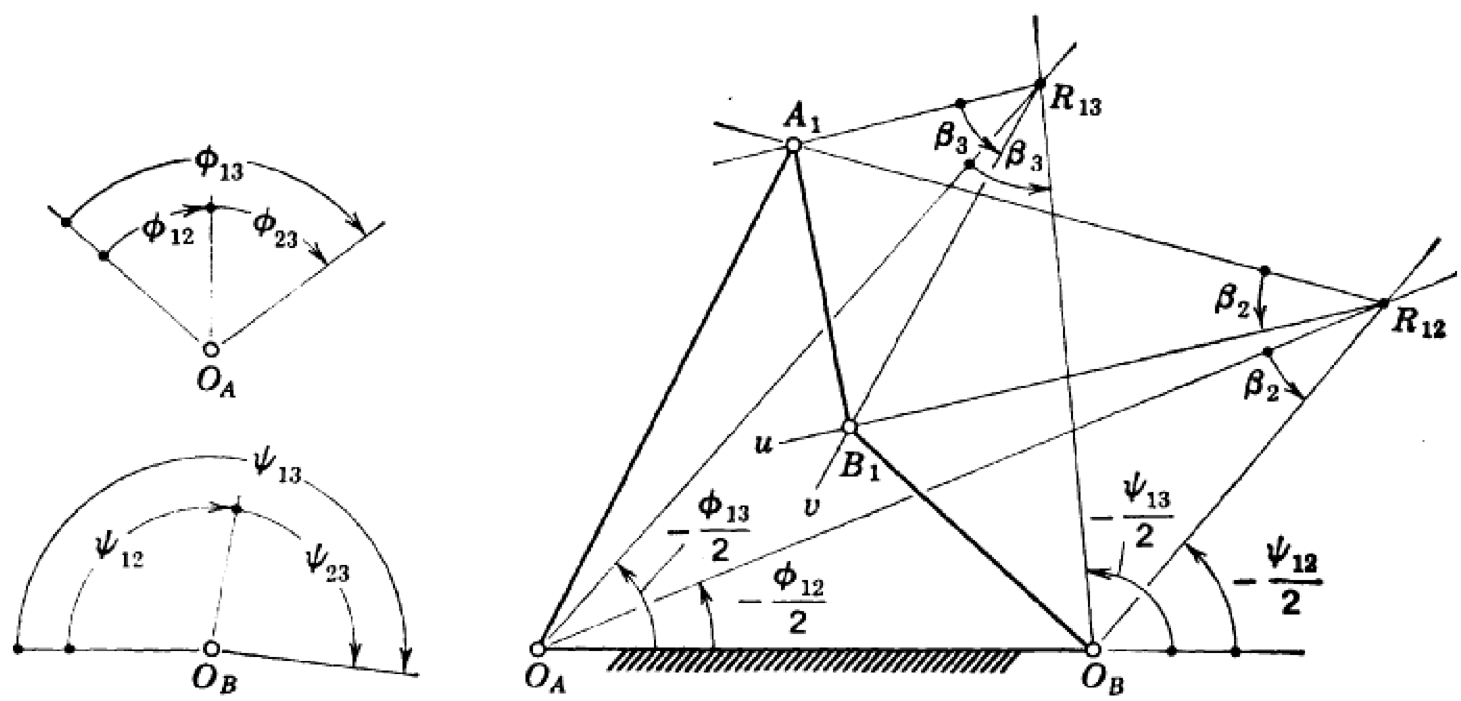

FIGURE 8-10 Solution of Prob. 5.

Problem 5 Design a four-bar linkage in which two successive clockwise angular displacements ϕ_{12} and ϕ_{23} of the crank produce, respectively, two successive clockwise angular displacements ψ_{12} and ψ_{23} of the follower (Fig. 8-10).

Solution Let $\phi_{13} = \phi_{12} + \phi_{23}$ and $\psi_{13} = \psi_{12} + \psi_{23}$.

1. Assume a convenient frame O_AO_B, and construct relative poles R_{12} and R_{13}.
2. Choose A_1 arbitrarily.
3. Draw the lines $R_{12}u$ and $R_{13}v$ such that

$$\beta_2 = A_1R_{12}u = O_AR_{12}O_B$$
$$\beta_3 = A_1R_{13}v = O_AR_{13}O_B$$

4. The intersection of $R_{12}u$ and $R_{13}v$ is B_1.

The linkage, chosen from among the available ∞^2 solutions (choice of point A_1), is a double-rocker mechanism. The choice of A_1, while good for displaying the construction, was unfortunate from an operational standpoint. An examination of the linkage, best afforded by a model, will uncover three shortcomings: (1) starting from position O_AA_1, the input link must first be turned counterclockwise, and then must have its direction of rotation reversed; (2) the linkage will go through a dead point when fully extended to the left; (3) it will be necessary to disconnect the linkage when it is fully extended to the right, move the links, and reconnect them before going to the final position. That the linkage does meet its specifications in a formal manner is small consolation for the roundabout way in which this is accomplished. The problem of how to remedy this situation will be enlarged upon in Sec. 8-5.

8-4 EXAMPLE: LOGARITHMIC-FUNCTION GENERATOR

If the output and input variables of a linkage are proportionally related to the variables of a *specified* function such as

$$y = f(x) \qquad \text{or} \qquad z = g(x, y)$$

the linkage is called a function generator. The linkage for $z = g(x, y)$ will obviously require two inputs, one for each of the independent variables x and y. In what follows, we shall not consider such double-input function generators: our attention will be directed to the simpler situation represented by $y = f(x)$ requiring only a single input.

The principle of a single-input function generator of the four-bar type is shown in Fig. 8-11. The independent variable x is to be represented mechanically by the rotation ϕ of the crank O_AA, or input, with the follower O_BB rotation ψ displaying the dependent variable y. The discrete relations between x and ϕ, y and ψ are usually made linear, but they need not be.

As an example, we shall design a four-bar linkage to generate the function $y = \log x$ in the interval $1 \leq x \leq 2$. The independent variable ranges from $x = 1 = x_s$ to $x = 2 = x_f$, or we may say that the range $\Delta x = x_f - x_s$ in general terms. The range of motion or angular sweep of the x pointer (link O_AA) corresponding to Δx will be designated $\Delta\phi = \phi_f - \phi_s$; this range is arbitrary and will be taken as 60° counterclockwise. On the assumption of linear relationships, any value of x within the interval of generation is related to its ϕ value by

$$\frac{\phi - \phi_s}{x - x_s} = \frac{\Delta\phi}{\Delta x} = r_x \qquad \text{or} \qquad \frac{\phi - \phi_s}{\Delta\phi} = \frac{x - x_s}{\Delta x}$$

The dependent variable ranges between $y_s = \log x_s = 0$ and

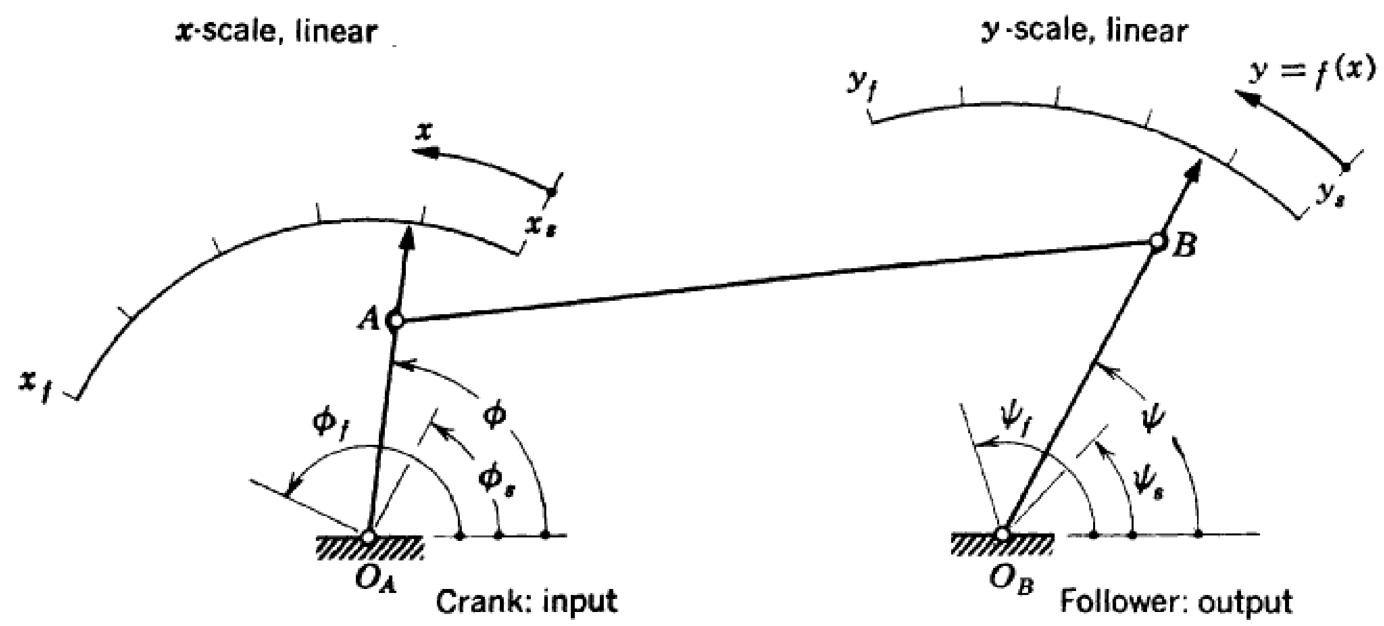

FIGURE 8-11 Principle of four-bar linkage function generator $y = f(x)$.

$y_f = \log x_f = 0.3010$, and the range is $\Delta y = y_f - y_s$. The corresponding range of motion of the y pointer is $\Delta\psi = \psi_f - \psi_s$; the actual value of $\Delta\psi$ is arbitrary and is chosen here as 90° counterclockwise. The variables y and ψ are related within the interval of generation through the linear relation

$$\frac{\psi - \psi_s}{y - y_s} = \frac{\Delta\psi}{\Delta y} = r_y \qquad \text{or} \qquad \frac{\psi - \psi_s}{\Delta\psi} = \frac{y - y_s}{\Delta y}$$

In this example, the four-bar linkage will be designed to give an exact value of the logarithmic function at only three points of the y-x curve, corresponding to three values of x at the accuracy points. Let x_1, x_2, x_3 be the accuracy points; they are chosen with Chebyshev spacing in the interval $1 \leq x \leq 2$, that is (Fig. 8-12),

$$\begin{aligned} x_1 &= 1.5 - 0.5 \cos 30° = 1.067 \\ x_2 &= 1.5 \\ x_3 &= 1.5 + 0.5 \cos 30° = 1.933 \end{aligned}$$

The corresponding values of y are

$$\begin{aligned} y_1 &= \log 1.067 = 0.0282 \\ y_2 &= \log 1.5 = 0.1761 \\ y_3 &= \log 1.933 = 0.2862 \end{aligned}$$

The change in x from the first to the second accuracy point is

$$x_{12} = x_2 - x_1 = 1.5 - 1.067 = 0.433$$

and from the first to the third accuracy point

$$x_{13} = x_3 - x_1 = 1.933 - 1.067 = 0.866$$

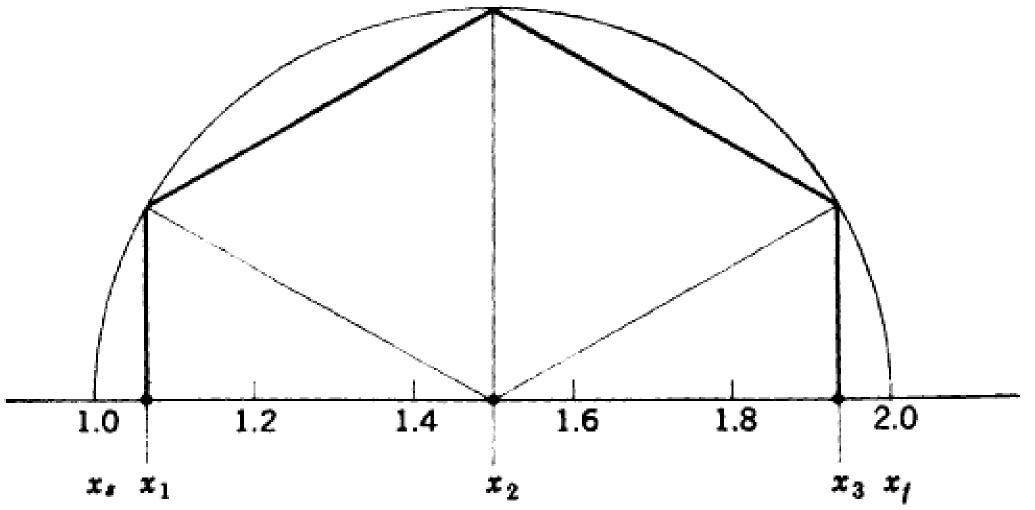

FIGURE 8-12 Three accuracy points with Chebyshev spacing in the interval $1 \leq x \leq 2$.

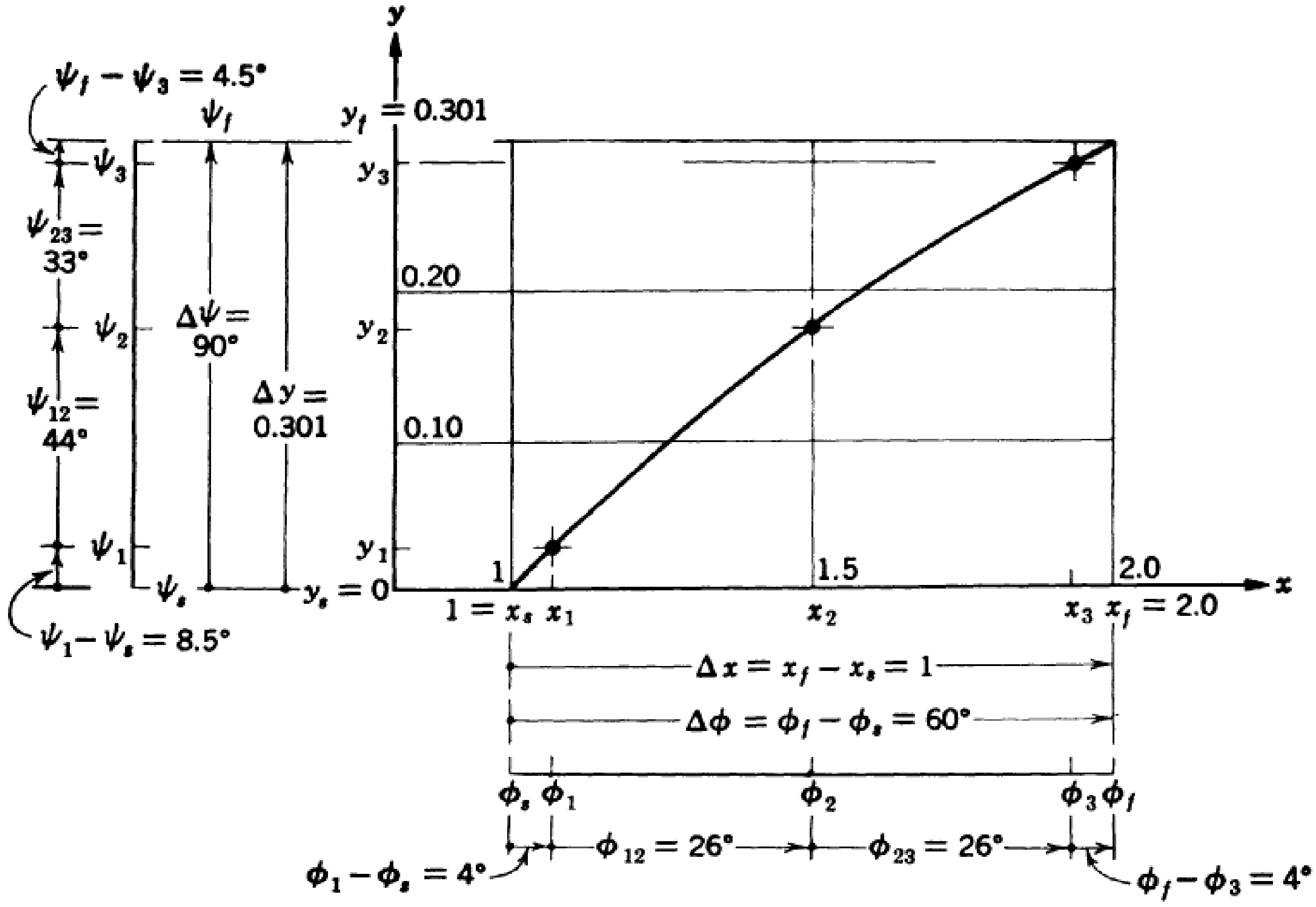

FIGURE 8-13 Function $y = \log x$, $1 \leq x \leq 2$ to be generated.

The corresponding changes in ϕ are therefore

$$\phi_{12} = \phi_2 - \phi_1 = \frac{x_2 - x_1}{\Delta x} \Delta\phi = \frac{0.433}{1} 60 = 26°$$

$$\phi_{13} = \phi_3 - \phi_1 = 52°$$

Similarly, the change in y from the first to the second accuracy point is

$$y_2 - y_1 = 0.1761 - 0.0282 = 0.1479$$

and from the first to the third accuracy point

$$y_3 - y_1 = 0.2862 - 0.0282 = 0.2580$$

The corresponding changes in ψ are

$$\psi_{12} = \psi_2 - \psi_1 = \frac{y_2 - y_1}{\Delta y} \Delta\psi = \frac{0.1479}{0.3010} 90 = 44°$$

$$\psi_{13} = \psi_3 - \psi_1 = 77°$$

All pertinent data are assembled in Fig. 8-13.

The problem is now reduced to the terms of Prob. 5, Sec. 8-3, and the solution proceeds as shown in Fig. 8-14. A frame length O_AO_B of 4 in. was chosen and the relative poles R_{12} and R_{13} constructed. An angle $\phi_1 = 90°$ and a crank length $O_AA = 3$ in. were selected, giving the point A_1 as shown. The lines $R_{12}u$ and $R_{13}v$ were drawn to give B_1 at their intersection. The desired linkage is $O_AA_1B_1O_B$ in the first position. Measurements from Fig. 8-14 yield

$$O_AA_1 = 3 \text{ in.} \qquad O_BB_1 = 1.68 \text{ in.} \qquad O_AO_B = 4 \text{ in.} \qquad A_1B_1 = 5.8 \text{ in.}$$
$$\psi_1 = 32°$$

From Table 8-1,

$$\phi_1 = \phi_s + 4 = 90° \qquad \text{or} \qquad \phi_s = 86°$$
$$\psi_1 = \psi_s + 8.5 = 32° \qquad \text{or} \qquad \psi_s = 23.5°$$

The linkage was next redrawn and fitted with ϕ and ψ scales (Fig. 8-15). A cursory examination shows that the follower rotations ψ_{12} and ψ_{23} will be produced as the input rotations ϕ_{12} and ϕ_{23} are imposed. Consideration of the link lengths shows the device to be a double-rocker linkage.

The performance of a linkage is gauged by how accurately it generates the specified function (Table 8-1). In Fig. 8-15, 11 values of ϕ were chosen at 6° intervals; the corresponding values of ψ were constructed and converted to y_{mech}. The error is the difference between $y = \log x$ and y_{mech}. The maximum value of the error in y was found to have a

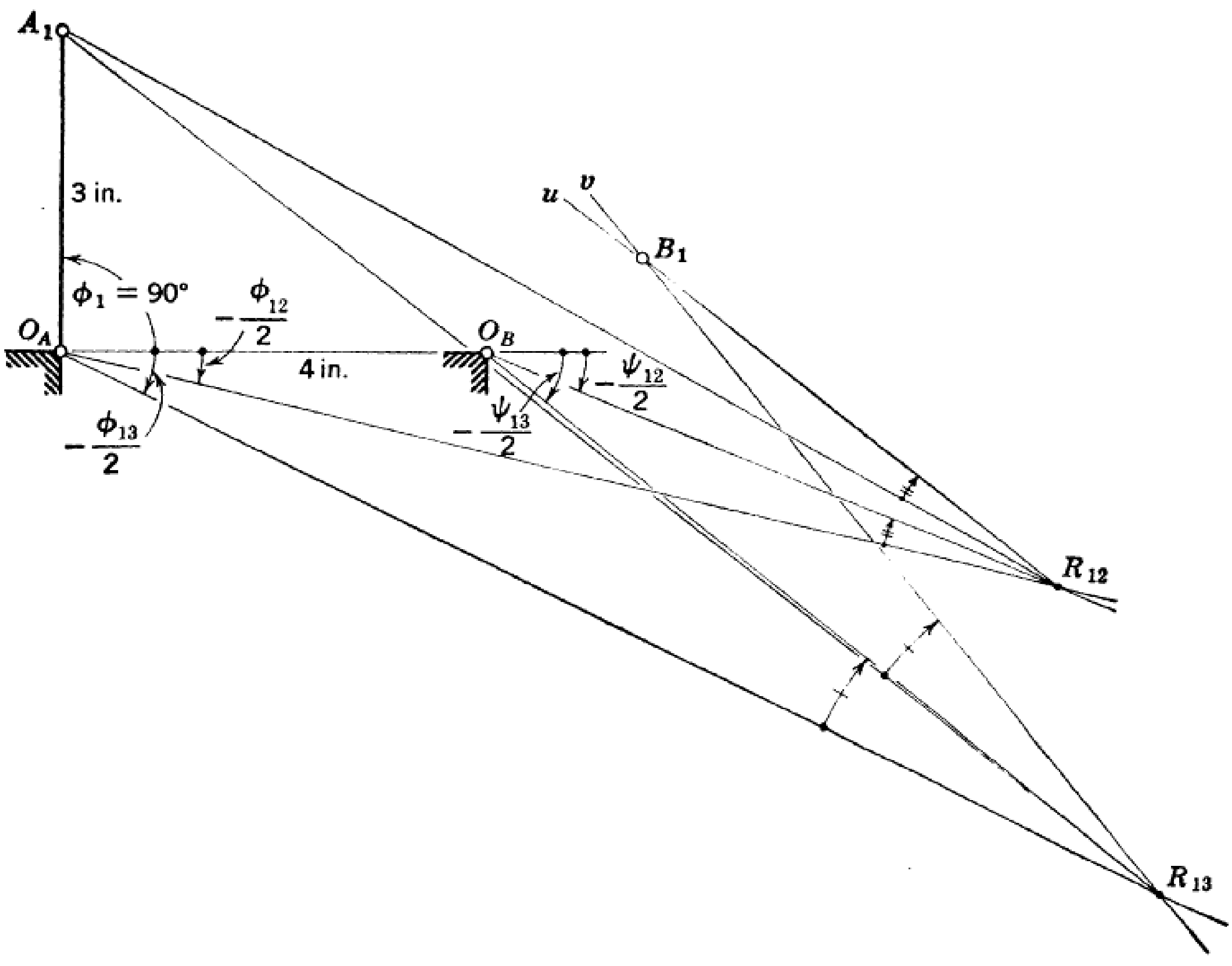

FIGURE 8-14 Synthesis of function generator $y = \log x$, $1 \leq x \leq 2$, three accuracy points.