CHAPTER X.

MECHANISMS INVOLVING NON-RIGID LINKS.

79. Non-rigid Links.—In giving a definition of a machine or of a mechanism we were careful to use the word "resistant" as applied to the material forming the links composing the mechanism. Many essential portions of actual machines are non-rigid, but are nevertheless resistant, and their occurrence, while it does little to complicate the machine from a kinematic point of view, often introduces dynamical problems of the greatest interest and difficulty. The different classes of non-rigid links, and pairs involving them, have already been noticed; we have now to study certain kinematic questions arising from their use.

In considering non-rigid links in mechanisms or machines it is necessary to take account of the way in which their form changes while in motion. One class of these links is composed of those which, while very yielding as far as bending or thrusting actions are concerned, do not change their length appreciably when a direct pull is applied. Belts, ropes, and chains, which come under this head, are therefore often of great use in machines where energy has to be transmitted in changing directions. This is usually done by causing the flexible tension-links, in the form of belts, ropes, or chains, to pair with, and communicate motion to, rotating drums or wheels. On account of their change of form, non-rigid links can have no virtual axes or virtual centres.

80. Velocity Ratio in Belt-gearing. Length of Belts.—The linear velocity of a rope or belt passing over two or more pulleys may be considered for kinematic purposes as being the same throughout its length. In practice the stretching of a rope or belt under load often has an appreciable effect on the velocity ratio of the pulley it drives; we shall here treat questions of velocity ratio as if the belt or rope were inextensible. Fig. 163 represents a pair of cylindrical pulleys connected by a belt, which may be "open" or "crossed" so that the pulleys rotate either in the same or in opposite senses. We shall for the present neglect the effect of the thickness of the belt or rope.

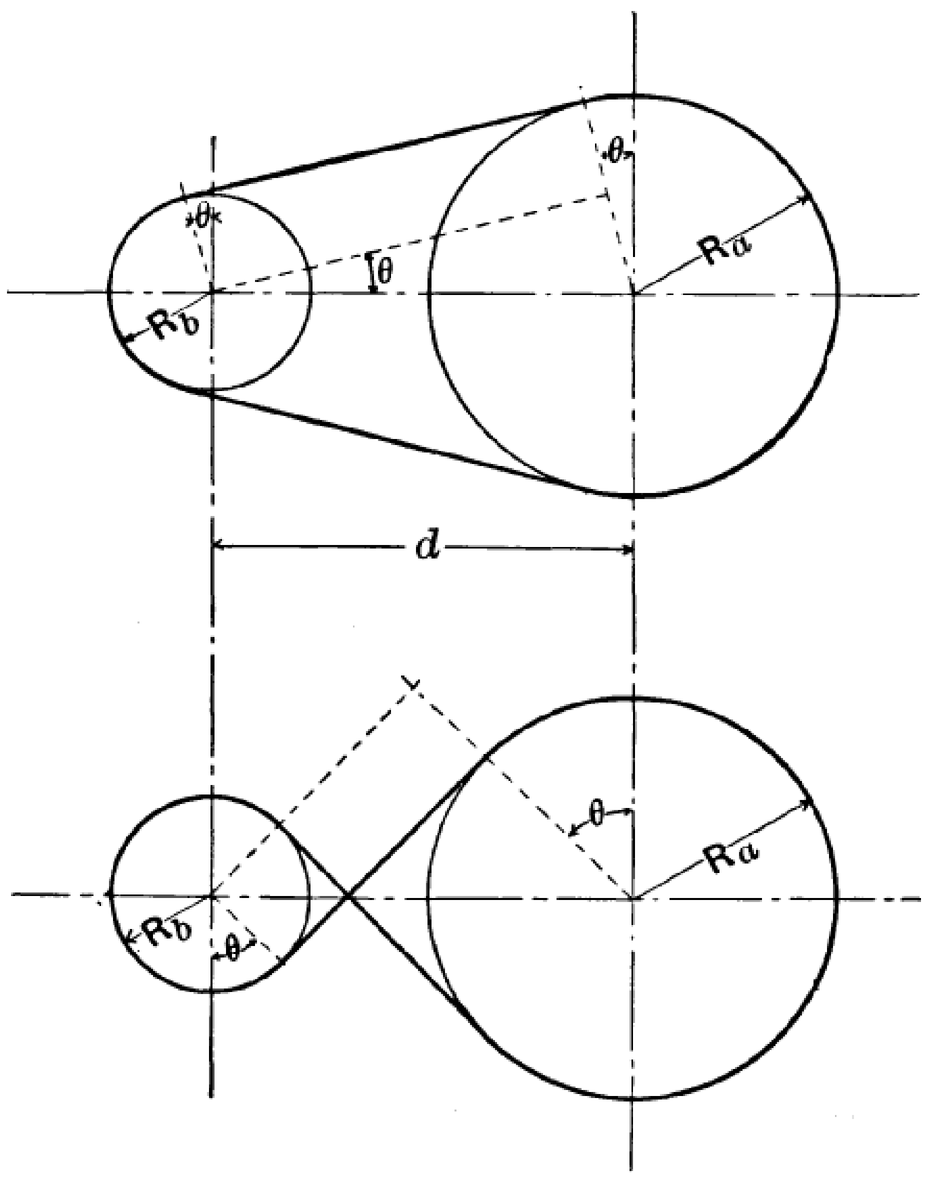

FIG. 163.

In these cases if V be the linear velocity of the belt and $R_a R_b$ the radii of the pulleys, the angular velocity ratio will evidently be found from the relation

$$\frac{\omega_a}{\omega_b} = \frac{V}{R_a} \cdot \frac{R_b}{V} = \pm \frac{R_b}{R_a},$$

the negative sign corresponding to the case of a crossed belt. It is, of course, assumed that there is no slipping.

The length of a belt is easily expressed in terms of the radii and the distance d between the centres of the pulleys. The total length of belt not in contact with the pulleys is

$$2\sqrt{d^2-(R_a \pm R_b)^2},$$

the negative sign here corresponding to the case of an open belt. If θ be the angle that the straight part of the belt makes with the centre line of the pulleys, then the length of belt in contact with the pulleys will be

$$(\pi+2\theta)(R_a+R_b) \text{ for a crossed belt}$$

and $$(\pi+2\theta)R_a+(\pi-2\theta)R_b,$$

or $$\pi(R_a+R_b)+2\theta(R_a-R_b) \text{ for an open belt,}$$

where $\theta=\sin^{-1}\dfrac{R_a \pm R_b}{d}$.

The expression for the total length of belt will then be for an open belt

$$2\sqrt{d^2-(R_a-R_b)^2}+\pi(R_a+R_b)+2(R_a-R_b)\sin^{-1}\frac{R_a-R_b}{d},$$

and for a crossed belt

$$2\sqrt{d^2-(R_a+R_b)^2}+(R_a+R_b)\left(\pi+2\sin^{-1}\frac{R_a+R_b}{d}\right).$$

It will be seen that the length of a crossed belt is thus constant so long as the sum of the radii and the distance between the centres of pulleys are constant quantities.

81. Belt-gearing for Variable Velocity Ratio. — Fig. 164 shows the arrangement of "cone pulleys" employed in driving machinery so as to render it possible, by shifting a belt from one pair of steps to another, to obtain at will any one of several velocity ratios. It is plain that the same *crossed* belt will run with the same tightness on any pair of steps so long as the sum of the radii of each pair is the same. An open belt, however, is generally required, in which case the tension will be different on each pair of steps, unless

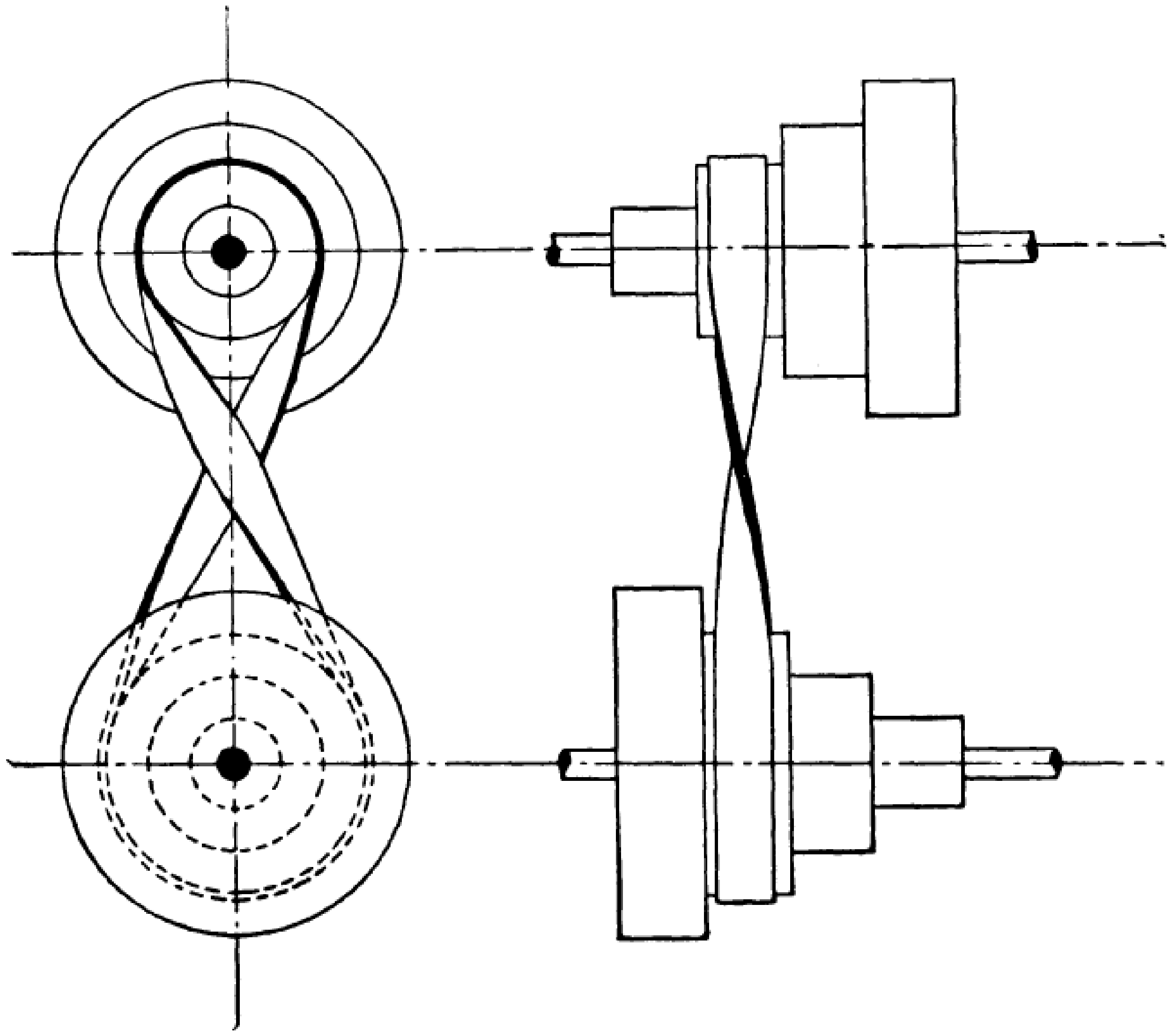

FIG. 164.

their diameters are specially calculated. Approximate methods for readily doing this have been devised,* while Reuleaux † gives a rigorous graphical treatment of the problem. Referring to Fig. 163, we have as an expression for the length of an open belt

$$l = 2\left[d \cos \theta + \frac{\pi}{2}(R_a + R_b) + \theta(R_a - R_b)\right].$$

* Unwin, Machine Design, Vol. I, p. 373; Smith, Trans. Am. Soc. M. E., Vol. X, p. 269.

† Reuleaux, The Constructor. Trans. by Suplee, p. 189.

Now $R_a - R_b = d \sin\theta$; therefore

$$l = 2\left(d\cos\theta + \frac{\pi}{2}(2R_a - d\sin\theta) + \theta d\sin\theta\right)$$

and

$$R_a = \frac{l}{2\pi} - \frac{d}{\pi}(\cos\theta + \theta\sin\theta) + \frac{d\sin\theta}{2}.$$

Similarly,

$$R_b = \frac{l}{2\pi} - \frac{d}{\pi}(\cos\theta + \theta\sin\theta) - \frac{d\sin\theta}{2}.$$

Take a pair of rectangular axes OA and OB, (Fig. 165) and make $OA = d$. Draw a curve CD, the involute of the circular arc AC, having O as its centre. Then, since the

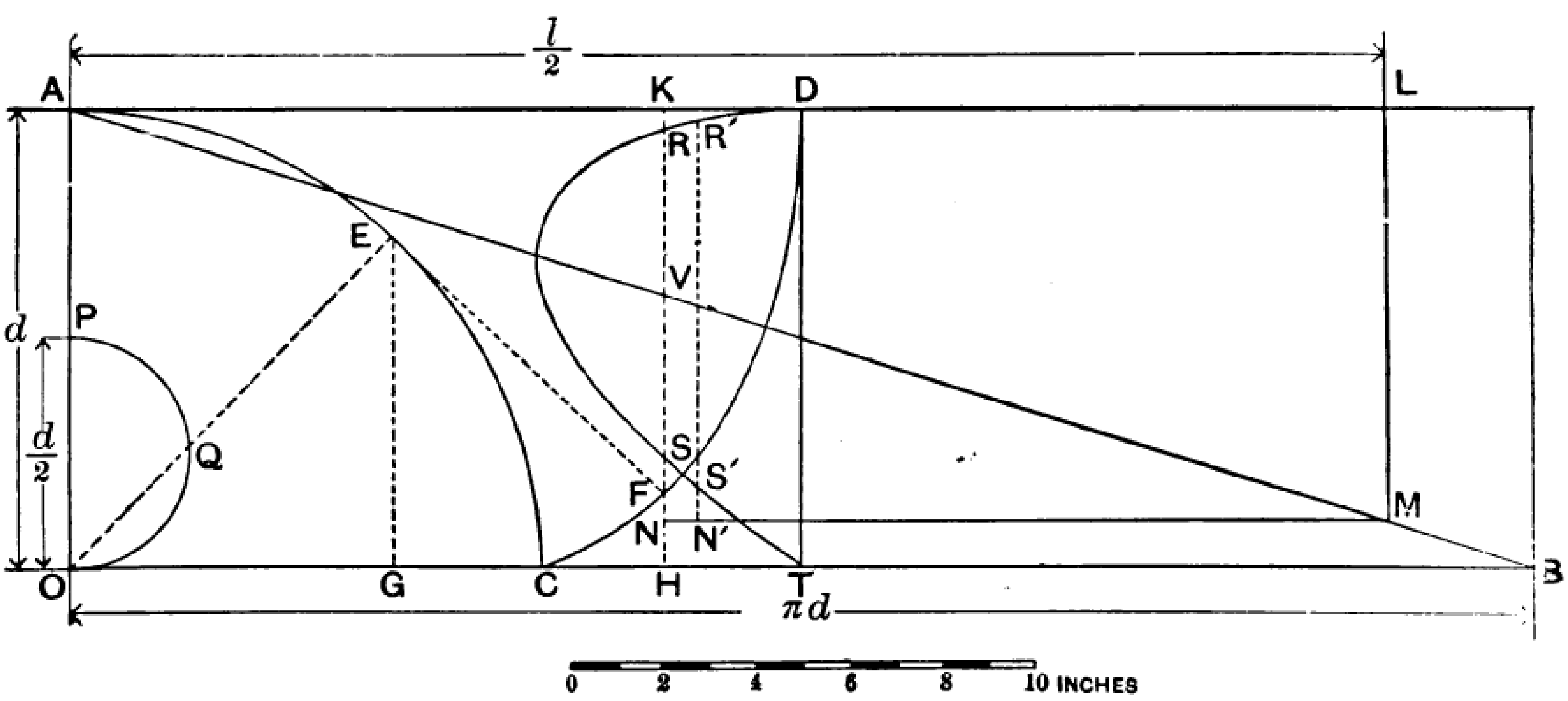

FIG. 165.

angle θ must lie between 0° and 90°, it must have some such value as COE, in which case the line EF, tangent to AC at E, and cutting the involute at F, has a length equal to the arc EC. Hence $EF = \theta d$, and, drawing KFH parallel to AO, we have $GH = EF \sin\theta$, and

$$OH = OG + GH$$
$$= d(\cos\theta + \theta\sin\theta).$$

Next make $OB = \pi d$ and join AB. Draw AD parallel to OB. Let HF meet AB in V and AD in K; then

$$\frac{KV}{OH} = \frac{AO}{OB} = \frac{1}{\pi},$$

and therefore $$KV = \frac{d}{\pi}(\cos\theta + \theta\sin\theta).$$

Again, if we set off $AL = \frac{l}{2}$, and draw LM parallel to AO, and cutting AB in M, we have

$$LM = \frac{l}{2\pi}.$$

Draw MN parallel to BO and cutting HK in N, then

$$\begin{aligned} VN &= KN - KV \\ &= \frac{l}{2\pi} - \frac{d}{\pi}(\cos\theta + \theta\sin\theta). \end{aligned}$$

To obtain the value of $\frac{d\sin\theta}{2}$ we need only draw a semicircle OQP having a diameter $\frac{d}{2}$; then

$$OQ = \frac{d\sin\theta}{2}.$$

Finally a curve $DRST$ may be drawn by setting off $VR = VS = OQ$, and repeating the construction as required. This gives

$$\begin{aligned} NR &= VN + VR \\ &= \frac{l}{2\pi} - \frac{d}{\pi}(\cos\theta + \theta\sin\theta) + \frac{d\sin\theta}{2} \\ &= R_a, \end{aligned}$$

and $$\begin{aligned} NS &= VN - VS \\ &= \frac{l}{2\pi} - \frac{d}{\pi}(\cos\theta + \theta\sin\theta) - \frac{d\sin\theta}{2} \\ &= R_b. \end{aligned}$$

Thus $$R_a - R_b = VR + VS = SR.$$

Plainly for given values of l and d we can determine R_a and R_b for any value of θ (or for any required velocity ratio) by the aid of the curve $DRST$.

In practice it is usual to find that the diameters of the first pair of steps or their radii, R_a and R_b, are given, together with d, the distance between centres of pulleys. The problem then is to find the radii of another pair of pulleys, R_a' and R_b', on which an open belt of the same length will run with a given velocity ratio $\frac{R_a'}{R_b'}$. The author has found the following a convenient method of utilizing the Reuleaux diagram for solving this problem, and for finding incidentally the length of belt required. This length, however, is not often necessary, as it is more easily measured from the pulleys when finished and in position.

Draw the rectangle AOB and the curve $DRST$ exactly as described above, and as shown in Fig. 165, making OA, say, 10 inches in length. This diagram can be used for finding pairs of radii of steps having any desired velocity ratio, and the lengths of these radii will be obtained in terms of d, the distance between the shaft centres. Having expressed R_a and R_b, the given pair of radii, in terms of d, it is easy, by applying a scale of inches and hundredths to the diagram, to determine that position of the line SR which will give the proper value to $R_a - R_b$. The length RN is then measured to the proper scale and the point N found. If required, the half length of the belt is then settled by drawing the line NM, and the next thing is to find another set of points R', S', N' such that $R'N'$ and $S'N'$ will have the ratio required for the radii of the next pair of steps. This is readily done by drawing on tracing-paper a set of radiating lines (Fig. 165a), VX, VY', VZ, arranged so as to cut all lines perpendicular to VZ in the required ratio, namely, $\frac{R'_a}{R'_b}$. It is convenient to draw another line, VY, such that lines perpendicular to VZ are also cut in

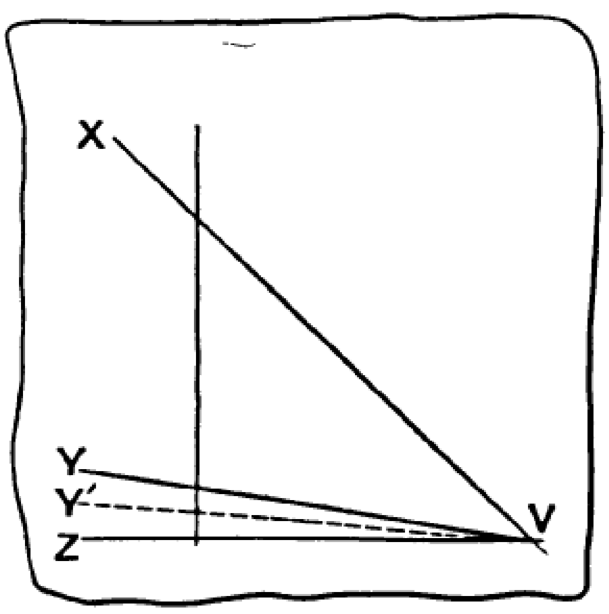

FIG. 165a.

the ratio $\frac{R_a}{R_b}$. By applying this diagram to Fig. 165, the three points R', S', N' can readily be pricked off in their proper positions. When measured to the proper scale, $R'N'$ and $S'N'$ give the values of the pair of radii required. In Fig. 165 the ratio $\frac{R_a'}{R_b'}$ is 12.0, while

$$\frac{R_a}{R_b} \text{ is } 6.0 \quad \text{and} \quad l = 5.66d.$$

If the real value of d is taken as 30 inches, while R_a and R_b are 25.2 and 4.2 inches respectively, the diagram gives for R_a' and R_b' the values 26.4 and 2.2 inches. An open belt of about 170 inches in length would run on either of these two pairs of pulleys.

It should be noted that when d is large in comparison with the size of the step pulleys, it is often sufficiently accurate to proportion the latter as if intended to run with a crossed belt; for this purpose the sum of the radii may be made constant.

To make allowance for the effect of the thickness of the belt or rope in our calculations it is only necessary to reflect that we have really taken the thickness of belt as being negligible when compared to the diameter of the pulley. In practice this is frequently not the case. Suppose, for example, that a belt whose thickness is a quarter of an inch is running on a pulley 6 inches in diameter. We assume that while passing round the pulley the layer of material at the centre of the thickness of the belt is neither stretched nor shortened, so that the arrangement will be equivalent kinematically to a pulley $6\frac{1}{4}$ inches in diameter on which a belt of negligible thickness is running. In other words, we take it that the effective radius of the pulley is in all cases to be measured to the centre of the thickness of the belt or rope.

82. Velocity Ratio in Chain- and Rope-gearing.—Rope- and chain-gearing is extensively used for the transmission of power, as well as in machinery for hoisting, winding, and lowering. In many cases it is necessary to provide the rope-drums or pulleys with guiding or retaining grooves. The various forms of rope and chain tackle are too familiar to require extended notice here; the ratio of the speed of the rope to the speed of the body moved by the tackle can always be readily found. As an example, we may take the Differential Pulley-block of Fig. 166. In this case the upper block has two sheaves a and a' rigidly connected or made in one piece; the chain is prevented from slipping on these sheaves by suitable projections in their grooves. Evidently on hauling in the sense shown by the arrow, the loop or bight of the chain passing around b will be shortened during each revolution of a and a' by an amount equal to the difference of the circumferences of those pulleys. Hence, if we call R_1 and R_2 the effective radii of a and a' we shall have

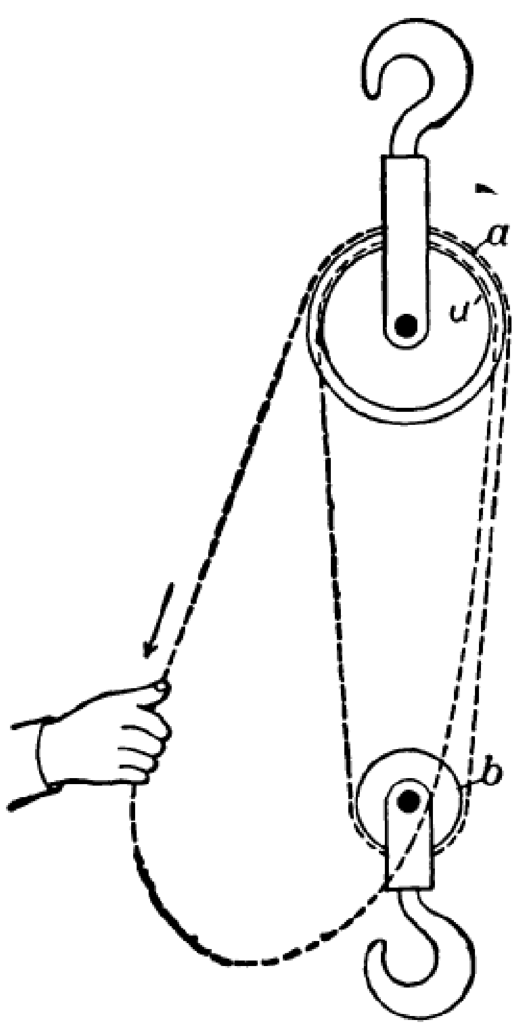

FIG. 166.

$$\frac{\text{speed of chain}}{\text{speed of hoisting}} = \frac{2\pi R_1}{\frac{1}{2}(2\pi R_1 - 2\pi R_2)}$$
$$= \frac{2R_1}{R_1 - R_2}.$$

Sometimes it is desirable to arrange hoisting gear in such a way that the velocity ratio is variable. For instance, in the winding gear of a deep mine it is necessary to wind the rope on a drum of continually increasing radius provided with a spiral groove, so that when one cage is at the bottom, its weight together with that of the attached rope may be balanced by the smaller weight of the other cage alone acting on a portion of the drum which is of larger radius.

A similar device is employed in the "fusee" of a chronometer.

In some cases shafts and pulleys are so connected by chain-gear that their velocity ratio is not uniform throughout the revolution. Fig. 167 shows one form of sprocket-wheel and chain. The wheel is furnished with teeth engaging with the links of the chain and effectually preventing slipping; these teeth should evidently have profiles composed of circular arcs parallel to the paths described by the centres of the pins as they move relatively to the wheel. On considering a pair of such wheels connected by a chain it will be seen that if their pitch-circles are of unequal diameters, their velocity

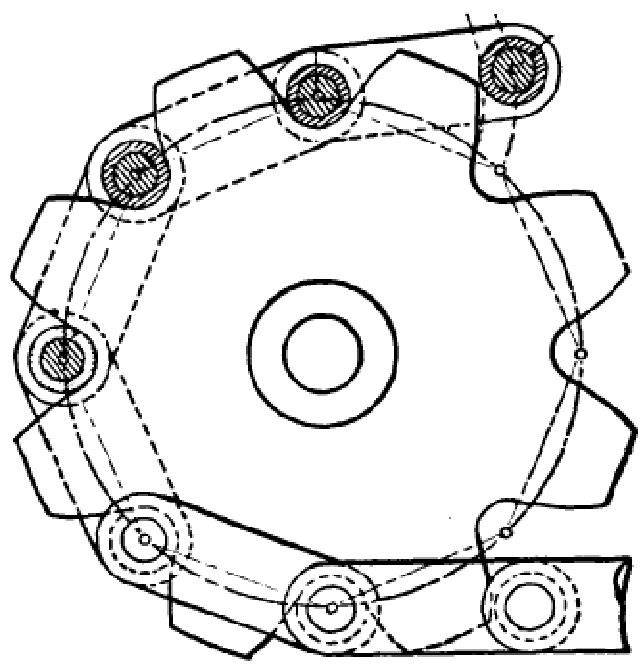

FIG. 167.

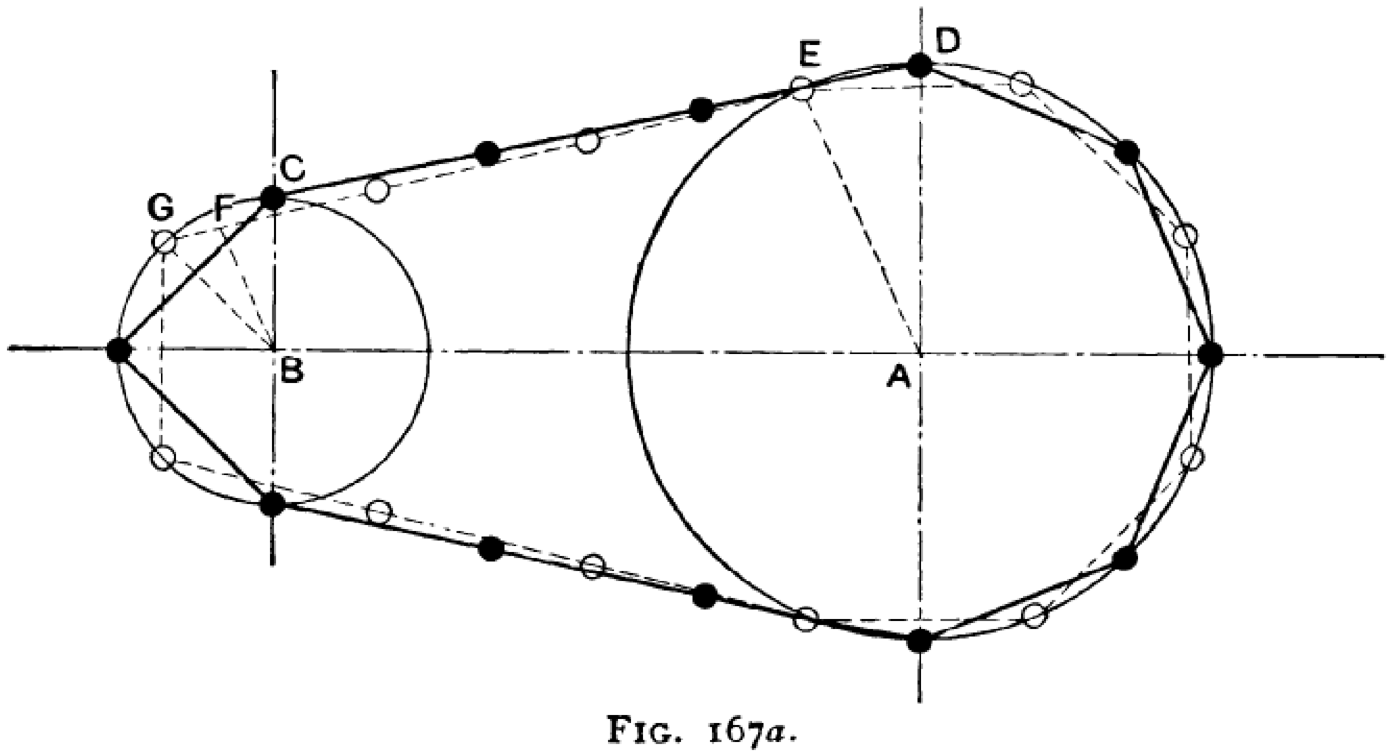

FIG. 167*a*.

ratio will not be the same in every position. Fig. 167*a* represents the centre line of a chain connecting a pair of sprocket-wheels; the wheels have four and eight teeth respectively. When in the position shown by full lines, the pair of wheels and the chain are equivalent to a four-bar mechanism or quadric crank chain *ABCD*. Applying the