CHAPTER VII.

CONSTRAINT AND VELOCITY RATIO IN HIGHER PAIRING INVOLVING PLANE MOTION.

60. Constraint of Bodies having Plane Motion.—It has already been stated that a body free to move in a plane possesses three degrees of freedom and has three degrees of constraint. Further constraint may be applied by causing such a body to touch certain points on the surface of a second rigid fixed body, these points being known as points of restraint. A *point of restraint* of a figure or body may be defined as a point on its outline, so touched by a point on the outline of a second fixed figure or body, that no relative sliding motion is possible along or parallel to the common normal to the two figures at the point of contact. When thus restrained the body or figure is considered as being kept in contact with the point or points of restraint.

We may take an example to illustrate the meaning of this definition, and to show the actual nature of points of restraint. Suppose (in Fig. 116*a*) that it is required to arrange a support or base, *a*, for a tripod, *b*, so that an instrument fixed on *b* can be removed from its support and replaced exactly in its previous position. This may be effected by providing *b* with three rounded points or legs, *CDE*. A hole, *F*, is made in the base, *a*, and is of pyramidal or conical form, so that if the rounded end of *C* is placed in *F*, there will be contact at three points of restraint; in this way, so long as the contact is maintained, the only possible relative motion of *c* and *a* will be one of rotation about some axis passing through the centre of the spherical surface of the end of *C*. The next step is to provide on *a* a slot or groove, *G*, of triangular cross-section as shown; when *D* is placed in this groove there will be two more points of restraint, and the only possible relative motion remaining will be a rotation about the axis *CD*. Finally the position of *b* is fixed relatively to *a* if the third point *E* is made to rest in contact with a flat surface, *H*, formed on or connected with *a*, thus furnishing the sixth point of restraint required (see § 7).

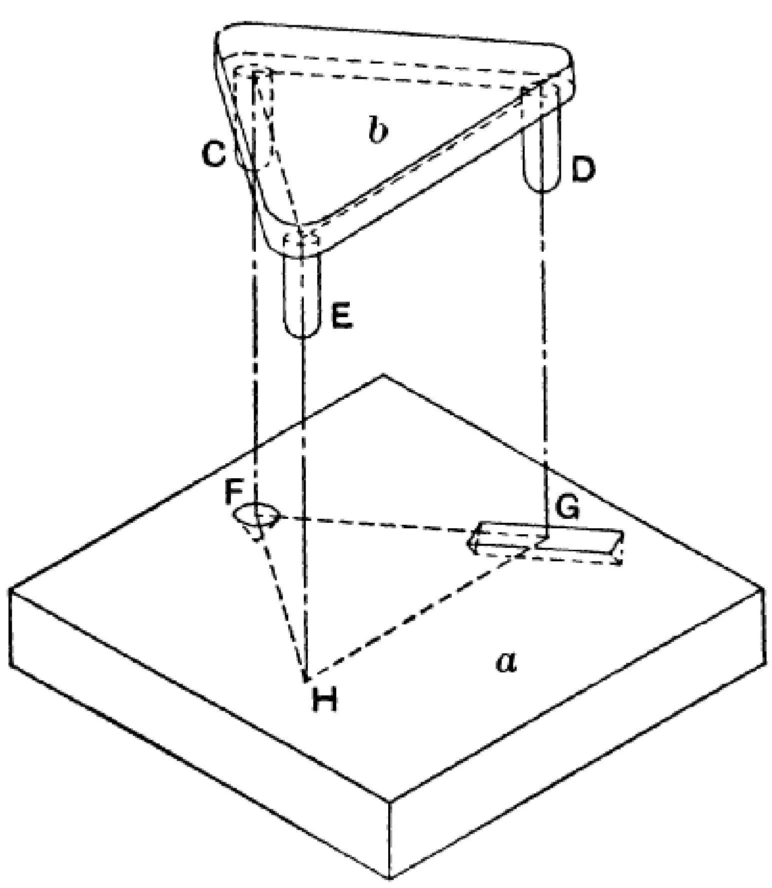

FIG. 116*a*.

The whole device is known as the "hole, slot, and plane."

The application of similar principles is illustrated in the design of Ewing's extensometer,* an instrument for measuring the deformation of test-pieces under stress. In this apparatus the bar or test-piece whose extension or compression is to be measured (*a* in Fig. 116*b*) carries a clip, *b*, attached by the points of two set-screws in such a way that *b* can move relatively to *a* about the axis of the set-screws at *B*. The clip *b* carries a projection, *b*′, ending in a rounded

* Ewing, Strength of Materials, p. 75.

point F. This point engages with a pyramidal or conical hole formed on a second clip, c, which is also secured to a by means of two set-screws at C. So long as F rests in its recess, b and c can have no relative motion unless the length BC alters; in that case the angular motion of b and c will be proportional to the extension or compression of a. Actually the projection b' is not rigidly attached to b, but can turn

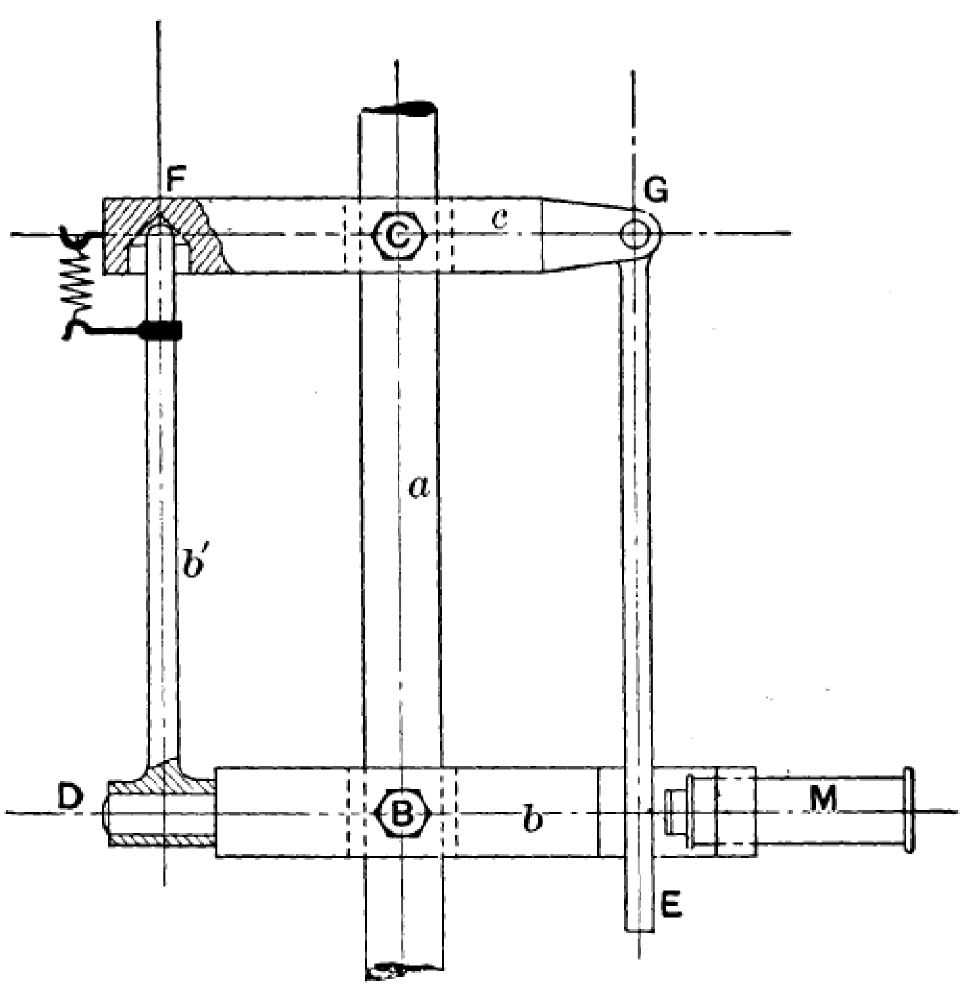

FIG. 116*b*.

through a small angle about the axis BD. This provision is made in order that any minute twist of the test-piece a about its axis BC may not affect the angular motion of b and c to any appreciable extent. This angular movement is indicated by the scale E attached to c; the distances CF, CG are equal, so that the movement of the scale, as read by the microscope at M, will be twice the actual deformation of the test-piece as taken on the length BC.

Similar methods are followed in designing the so-called kinematic clamps and kinematic slides.*

A *kinematic clamp* is a contrivance intended to fix completely the position of one body with reference to another; a *kinematic slide* permits one body to have one degree of freedom with reference to another.

On consideration it is plain that in a kinematic clamp or slide the points of restraint must be suitably placed with regard to the shape of the body to be restrained.

It is thus proper to inquire what must be the disposition of the points of restraint required, either to define the position of one body relatively to another, or to permit the movable body to retain one degree of freedom, and thus to constrain its motion completely. We shall suppose that the movable body at first possesses three degrees of freedom, and is capable of plane motion.

Let a (Fig. 117) be such a body, and let a fourth point

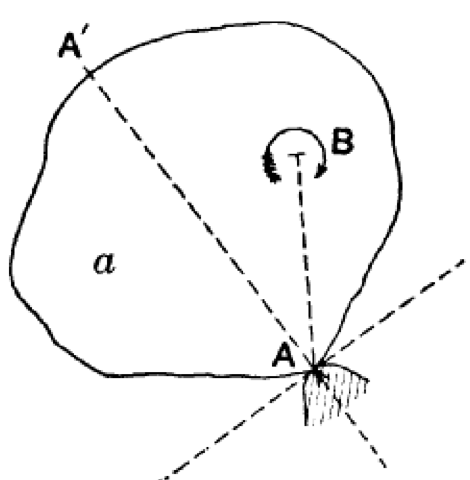

FIG. 117.

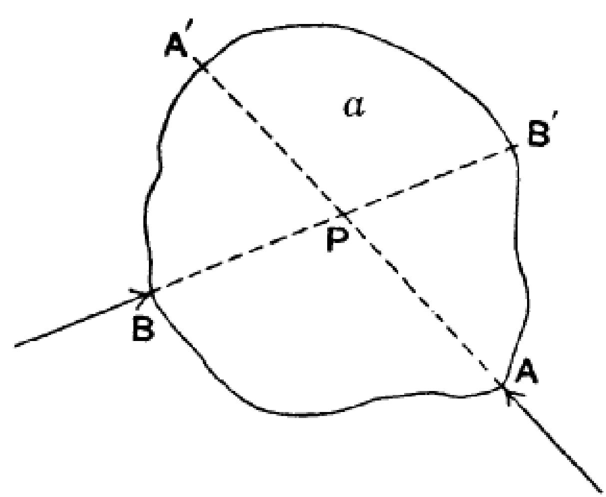

FIG. 118.

of restraint, A, be provided, in addition to the three points necessary for insuring plane motion. The arrow-head then represents the fourth point of contact of the restraining or fixed body.

Draw AA' normal to the tangent of the outline of a at A.

Any possible motion of a may be regarded as an instan-

* For an example of a kinematic slide, see Min. Proc. Inst. C. E., Vol. CXXXII, p. 49.

taneous rotation about a virtual axis perpendicular to the plane of motion. We need therefore only consider how the single point of restraint, A, affects the possibility of turning the body a about such an axis. It must be remembered that by the definition of a point of restraint, a is to be kept in contact with the restraining body. This is impossible if the virtual centre is not somewhere along AA', for if the virtual centre were, say, at B, a point about which only right-handed rotation is possible, it is plain that such rotation could only occur if the point A ceased to touch the restraining body. Hence we see that any possible instantaneous motion of a must be about a virtual centre situated in AA', and any motion of translation must be along a line at right angles to AA'.

Next consider the effect of keeping the body a in contact with a restraining body at two points, A and B. Let the normals AA', BB' intersect at P. The body is then only capable of an instantaneous rotation about P. If the normals are parallel, then only an instantaneous motion of translation, i.e., rotation about an infinitely distant axis, will be possible.

On adding another point of restraint, C (Fig. 119), it will be found that if we suppose that the body a remains in contact with the three new points of restraint, A, B, C, no movement is possible, except when the three normals intersect in one point or are parallel. In these cases instantaneous turning about the point of intersection, and instantaneous translation about an infinitely distant axis, are respectively possible, so that a at the instant considered will thus possess one degree of freedom and will have constrained motion.

In Fig. 119 a little consideration shows that no movement at all is possible except about an axis situated within the triangle PQR, so long as the restraining body is rigid. The whole field of motion, with the exception of PQR, then becomes what Reuleaux calls a "field of restraint." But if movement did occur about an axis placed within the triangle PQR (in the figure such rotation could only be right-handed), the body a would at once cease to touch the restraining points with which we suppose it to be kept in contact. A similar result will be found with other arrangements of the points of restraint, and therefore in general

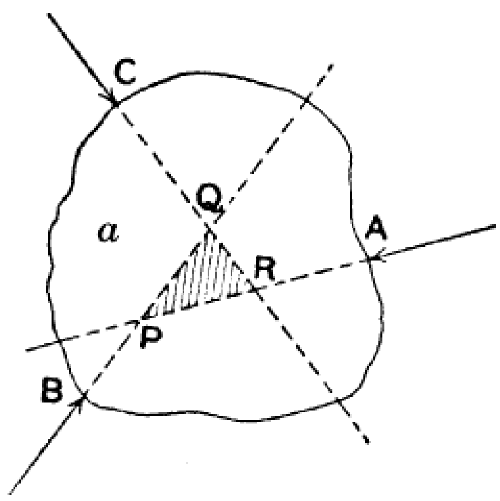

FIG. 119.

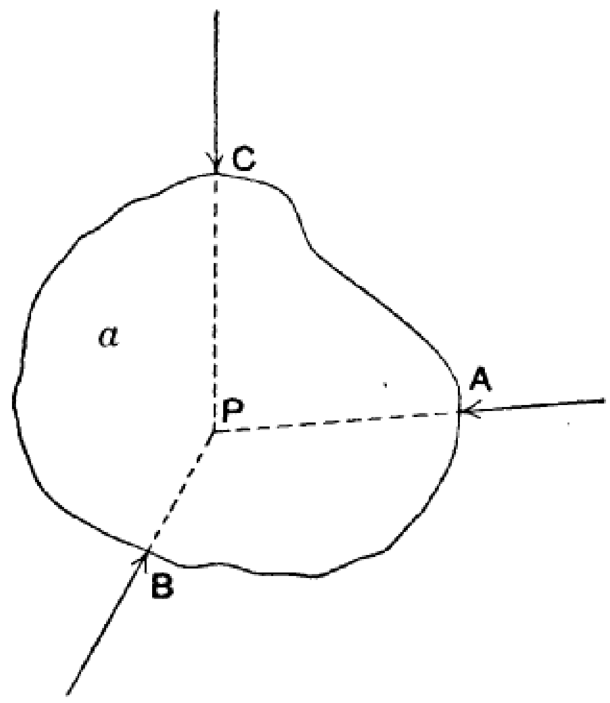

FIG. 120.

the position of a will be fixed if it is made to touch the restraining body at the three additional points A, B, C, a result already stated in § 7.

Fig. 120 shows the case in which the three normals meet in a point, P. If the shape of the body a is such that no point of restraint can be so applied as to have a normal that does not pass through P, then the body cannot be fixed by the application of three or any number of points of restraint, and its shape must be altered for that purpose. For example, a circular disc having plane motion could not be so fixed.

It is thus evident (a) that if one of two rigid bodies capable of relative plane motion remains in continuous contact with two points of restraint formed on the second body, the relative motion is constrained, and the virtual centre of the two bodies is always at the intersection of the two common normals.

Also (*b*) if three points of restraint are employed, and contact at all three is continuous, constrained relative motion is only possible if the three common normals intersect in a point, or are parallel.

The reader will find that the constraint of the motion of a body by means of such points of restraint as have been defined above is an easier matter than the limitation of the movement of a body by points of contact with a second fixed body, if no force is supposed to keep the two bodies in contact. In this case the bodies would possess greater freedom of motion than under the restrictions we have supposed. The theory of constraint has been treated by Reuleaux * and by Burmester,† to whose works the student is referred for information on the subject.

61. Closed Higher Pairs having Plane Motion.—Let us next suppose that the moving body *a* and the second or fixed body *b*, while kept in continuous contact, have such forms that one is the geometrical envelope of the other, and that in every position the normals at the several points of contact are either parallel or meet in a point. It is obvious that in this case at any instant *a* can move in one way, and in one way only, with reference to *b*; in other words, *a* and *b* will form a closed pair. We proceed to consider some examples of such pairing, which in general will be higher pairing, in accordance with the definitions in § 2.

In Fig. 121*a*, let *ABCD* be a figure (called by Reuleaux a Duangle), drawn by describing the arcs *ABC*, *CDA*, with a radius equal to *BD*, and with *D* and *B* as centres respectively. Suppose that this figure, representing a body, *a*, having plane motion, is made to touch two lines, *PQ*, *QR*, inclined at an angle of 60°, the points *E* and *F* on these lines forming points of restraint for the duangle, and the lines *PQ*, *QR* representing the profile of the restraining body *b*. The normals at *E* and *F* to *QR* and *QP* will inter-

* Reuleaux, Kinematics. Chapter III. † Burmester, Kinematik, Chapter V.

sect at O, where they make an angle $FOE = 120°$, and they must pass respectively through the points B and D, since these points are the centres of the arcs ADC, ABC.

As the duangle moves in contact with PQ and QR, the

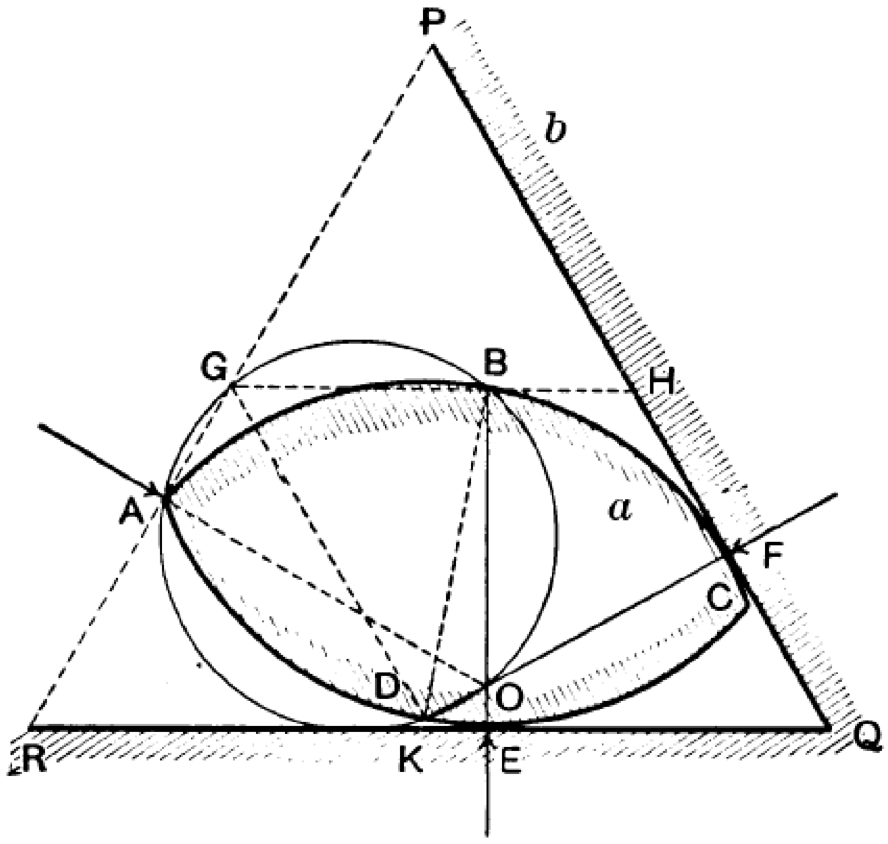

FIG. 121*a*.

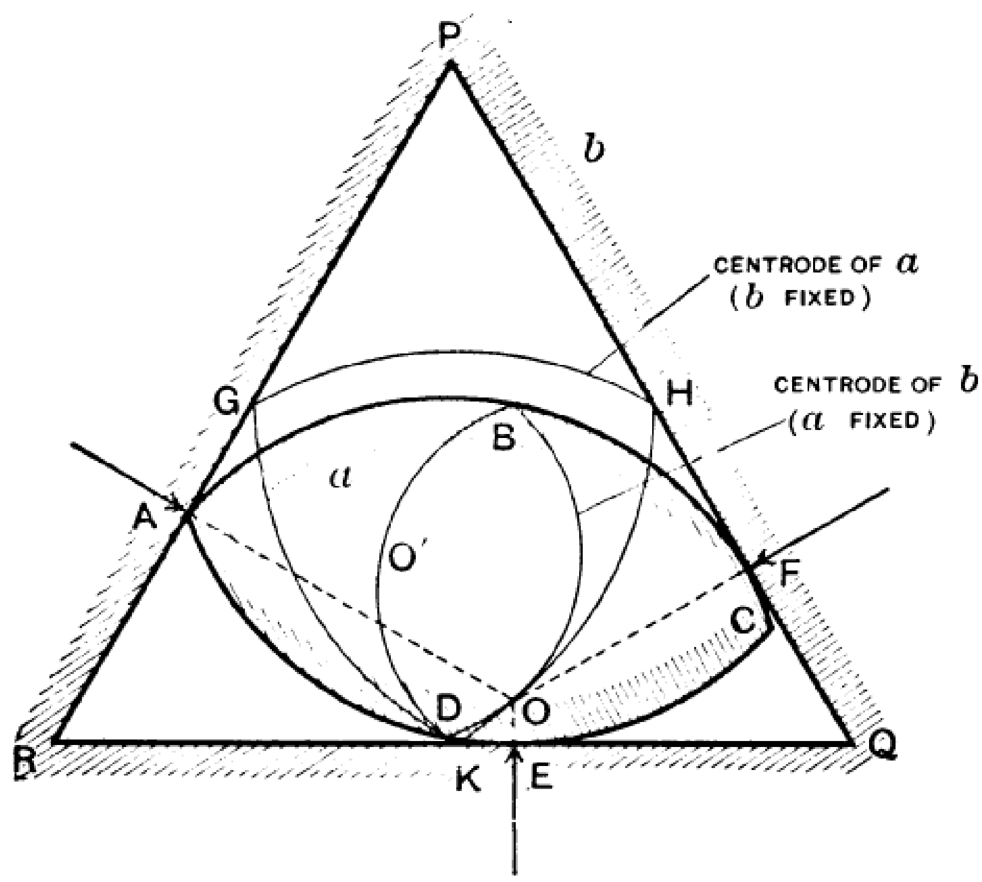

FIG. 121*b*.

path of B must be a straight line, GBH, parallel to QR and at a distance, BE, from it. The path of D similarly must be a line, GDK, parallel to QP. Hence the motion of the duangle relatively to PQ, QR will be the same as that of a straight line of constant length, BD, whose ends lie continually upon two lines, KG, HG, enclosing an angle of 60°; further, the virtual centre of the two bodies will be the point O, the intersection of the two common normals.

Since the angles OBG, ODG are right angles, a circle may be drawn on GO as diameter, passing through the points GB, OD. The point A also lies on this circle, since the angle BAD is 60°. Join AG. Then so long as the curves ABC, ADC touch the lines QP, QR respectively, the angle $AGD =$ angle $ABD =$ constant. Thus A lies continually on a line, RP, drawn through G and inclined at 60° to RQ. PQR is then an equilateral triangle, inside of which the duangle moves. The relative motion of the triangle and the duangle will be constrained if AO is the normal to PR at A; i.e., if the three normals at the points of contact meet at O. This is seen to be the case, for the angles AOD, ABD, ADB, AOB are all equal. Hence AO bisects the angle BOD and is perpendicular to PR.

The path described by O with reference to the triangle PQR is the centrode of the duangle. It evidently consists of a curve joining K and H. Now in any position the circle drawn on GO as diameter and passing through B and D has a chord, BD, of constant length, and the angle BGD is constant. Hence GO, the diameter of this circle, is the same (length $= GH$) for all positions of O. Thus O lies on a circular arc joining K and H and having G as centre, and the complete locus of O with regard to the triangle is an equilateral curve-triangle GKH (Fig 121*b*). Since the angle BOD is constant, the locus of O with regard to the duangle is seen to be a duangle $BODO'$, the radius of whose sides is $\frac{1}{2}GO$.

The whole relative motion of the duangle $ABCD$ and the triangle PQR is thus represented by the rolling of the duangle $BODO'$ inside the curve-triangle $GHOK$. The centrode of $ABCD$ with regard to the triangle PQR is $GHOK$;

that of the triangle with reference to $ABCD$ being $BODO'$. Any point on the duangle $ABCD$ will have a path made up of trochoidal curves described on the plane of the triangle PQR, and *vice versa*.

Relative motion of the duangle and the equilateral triangle may evidently be represented by the rolling together of a pair of circular arcs, one having a radius twice that of the other. Points on either figure will therefore describe trochoidal curves on the other.

The example just given will indicate the method of studying the relative motions of the elements of higher pairs having plane motion. A large number of closed higher pairs may be devised by utilizing figures of constant breadth. The equilateral curve-triangle previously mentioned is such a figure, and its motion relatively to a circumscribed square may be followed as an exercise.

A number of other forms are given by Reuleaux in the chapter already quoted. The student should note in all these cases that the form of the path described on b by a point on a is not the same as that described on a by the corresponding point on b, a condition previously mentioned as being characteristic of higher pairing.

62. Form of Elements for a Given Motion.—Having illustrated the method of determining the centrodes and the relative motion in the case of higher pairs of mutually restraining elements of given profile, we have next to show how to solve the converse of this problem, namely, how to find the forms of a pair of elements whose relative motion is previously decided. The relative motion in question must, of course, be defined by the forms of a pair of given centrodes, the mutual rolling of which, as already stated, represents the relative motion required. It most frequently happens in practice that we have also given the form or profile of one element of the pair, and the form of the second has to be found.

Let *AA* and *BB* (Fig. 122) be a pair of centrodes, of which *A* belongs to, or is traced upon, a body whose profile is *aa′*. It is required to find the profile of a second body to

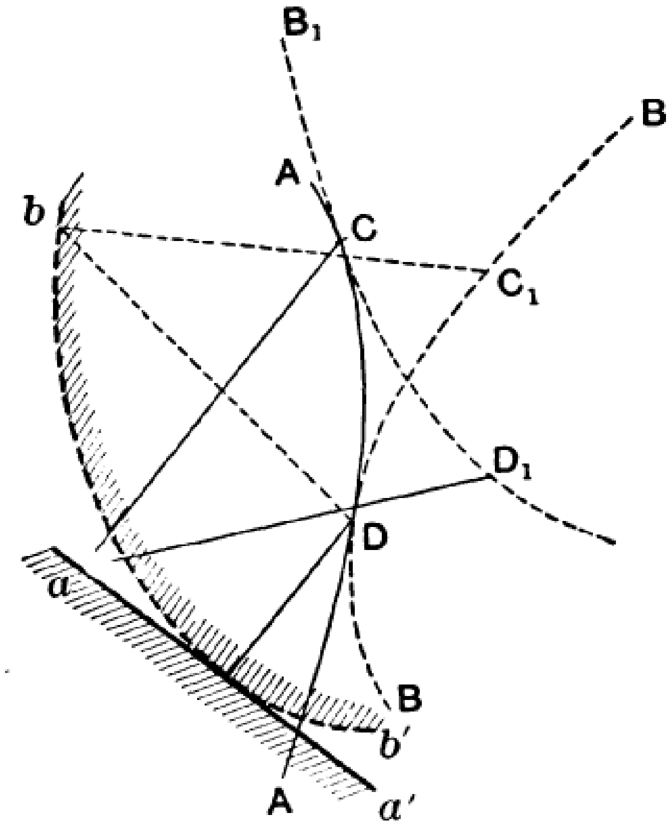

FIG. 122.

which the centrode *BB* belongs; the profile to be such that while the two bodies remain in continuous contact the centrodes will roll on one another and the bodies will thus have the desired relative motion.

Take any point, *a*, in the profile *aa′* and draw *aC* normal to *aa′* at *a*, and cutting the centrode *A* at *C*. In this case for convenience *aa′* is shown in the figure as a straight line, but it may, of course, be of any form.

At the instant when the profile *bb′* (to be found) touches the given profile at the point *a*, *aC* must be the common normal, and the virtual centre of the two bodies must lie on this normal, for otherwise contact would not be continuous. The point *C*, where the normal at *a* cuts the centrode *A*, must at that instant be the virtual centre of *bb′* with regard to *aa′*, since the curve *AC* is the locus of the virtual centre of *b*. *AC* may be regarded as being attached to *aa′*, since it is a curve traced on the body represented in outline by *aa′*. We proceed to find a point, *b*, on the profile of the second or moving body, such that when a and b are in contact C is the virtual centre of the two bodies and aC the common normal.

Suppose that the centrodes AA and BB are in contact at some point D, and measure along the centrode BB a length DC_1 equal to the length of DC measured along AA. Draw B_1CD_1, representing the centrode B in the position it occupies when a is the point of contact of the two bodies and C is their virtual centre, and make $CD_1 = CD$. Join aD_1.

Then since the outline of bb' may be regarded as attached to the centrode B, any point on that outline having the same position in relation to C_1 and D that the point a has in relation to C and D_1 will be the point that touches a when the centrodes touch at C. Accordingly we need only make $bD = aD_1$ and $bC_1 = aC$ in order to determine the position of b. The point b is then a point on the required profile which will touch the point a when C is the virtual centre of the two bodies. In the same way we can determine any other point on the profile required, and it only remains to provide the resulting body with the restraint required to prevent any other motion than that desired. This would in general be done by so forming the body bb' that it possesses at any instant three points of contact with aa', the normals to these points always intersecting at the virtual centre. It would, in fact, be necessary to repeat the construction of Fig. 122, assuming two other portions of the outline of aa', and finding two new portions of the outline of bb', the centrodes, of course, remaining the same as before. It may be noted that while the relative motion of the centrodes is one of simple rolling, that of the two outlines is in general rolling and sliding combined.

63. Condition for Uniform Velocity Ratio.—We have seen in § 60 that when two bodies are in continuous contact and are capable of constrained relative motion, the normals at the points of contact must intersect at the virtual centre.

Consider now the case of three bodies (Fig. 123) of which