CHAPTER V.

DETERMINATION OF VELOCITY AND ACCELERATION IN PLANE MECHANISMS.

48. Velocity and Acceleration Determined from Virtual Centres.—It is often necessary to determine the magnitude and direction of the velocity or acceleration of a given point of a given link in a plane mechanism. Such a calculation, for example, is frequently required if we wish to find the forces acting on a part of a machine when in motion, with a view to the correct proportioning of such a part to the work it has to do.

We have already studied this problem in certain cases, especially as regards the cross-head of a direct-acting steam-engine; the question has now to be discussed in a more general manner.

In a given mechanism, having given the velocity of a point on one link, and having found the positions of the various virtual centres, it is possible to determine the velocity of any point on any one of the links.

Take for example the beam-engine of Fig. 96, in which we suppose V_c, the velocity of the crank-pin, to be known. It is required to find the actual linear velocity (i.e., the velocity with relation to the frame or fixed link) of the piston and rod *b*.

Let *a* be the fixed link, *b* the piston, *d* the beam, *e* the connecting-rod, and *f* the crank.

First find O_{bd} at the intersection of a horizontal line through the beam centre O_{da} and the line joining O_{bc} and O_{cd}. Note that O_{ba} is at an infinite distance. Next find

O_{df}, draw a horizontal line through O_{af}, and find its intersection with the line joining O_{db} and O_{df}. This point is O_{bf}.

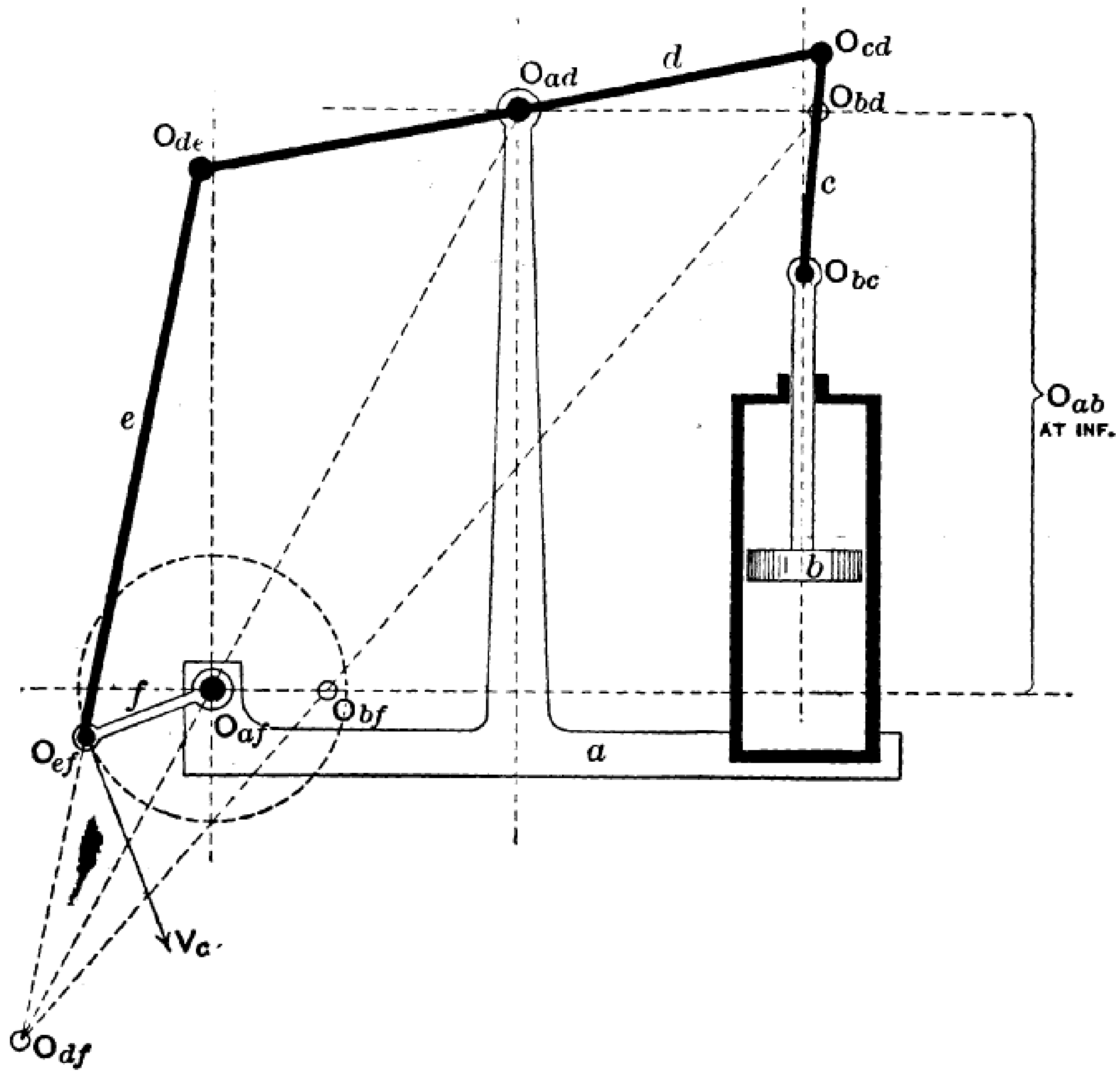

FIG. 96.

All these centres are readily found, remembering that they lie in threes in straight lines.

The point O_{bf} is a point common for the instant to the links b and f. Let the length of crank $=r$ and let the distance $O_{af} \ldots O_{bf}=m$.

Then the actual linear velocity of the point O_{bf} (considered as a point on the link f) must be $V_c \times \frac{m}{r}$, in a direction perpendicular to the line $O_{af}O_{bf}$, and this must also be the velocity of the link b, since O_{bf} is for the instant a point on that link also. The same construction will give the velocity of the piston for any position of the mechanism, except when the crank is on the dead-centre.

A similar method may be used in any case in which the various virtual centres can be found, but is not always possible for all positions of the mechanism, because many of the centres periodically recede to an infinite distance. This fact considerably reduces its practical usefulness.

49. **Method by Using Point-paths.**—The velocity of a given point on any link may be most simply determined for any given position of a mechanism by carefully drawing (to as large a scale as possible) the mechanism in two positions, one slightly before and the other slightly after the given position. The velocity of some one point of the mechanism being known, the velocity of the given point is readily found by comparing the displacements of the two points in the short time supposed to elapse between the two positions drawn, the direction of motion being known from the point-path on the drawing. It should be noted that this method of finding the velocity required is not susceptible of great accuracy, because the displacements whose ratio is measured must be supposed very small, in order that the result obtained may be as nearly as possible the true velocity of the point when the mechanism is actually in the given position. Hence the ratio of the displacements is difficult to measure. The method is nevertheless often used in practice.

As an example the mechanism of Fig. 97 may be taken. The figure shows Bremme's valve-gear.* It consists essentially of a lever-crank chain, the motion of the valve being taken from a point on one link produced. The figure, necessarily drawn here to a small scale, shows the proportions of an actual gear for a small marine engine. The eccentric of the engine corresponds to the crank of the lever-crank chain, and in practice coincides in angular position on the shaft with the engine-crank. The dimensions are:

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* See *Mechanical World*, September 2, 1889.

$AC = 1\frac{3}{4}''$ (throw of eccentric);
$CE = 10\frac{1}{2}''$;
$CD = 15\frac{3}{4}''$;
$EB = 14''$;
$AB = 20''$ (when the engine is going ahead).

The engine is reversed by altering the position of the suspension point B, as shown by the dotted arc.

It is required to determine the vertical component of the velocity of the point D (from which the valve is driven by a long rod) for any position of the gear, supposing the eccentric AC rotates uniformly at a speed of 170 revolutions per minute.

We first take a number of positions of AC (in this case 12 in one revolution), corresponding to equal small intervals of time (in this case 0.0294 second), and the corresponding positions of E and D are found. They are shown on the diagram and numbered successively, those of D forming points on a closed curve roughly oval in shape.

The vertical displacements of the point D have been plotted on a time base, giving the diagram xx. From this the velocity curve YYY has been drawn by the method of § 18. On determining the scale of the diagram we see that between the positions 4 and 5 the valve moved 0.60 inch upwards in $\frac{1}{12}$ revolution, i.e. in 0.0294 second. The velocity of the valve at the middle of that interval will therefore be approximately

$$\frac{0.60}{12 \times 0.0294} = 1.70 \text{ foot per second.}$$

In the same way the maximum downward velocity of the valve is found to be while the crank is moving from 10 to 11, and its value is about 2.00 feet per second.

If necessary the vertical *acceleration* of the valve can be determined as in § 19.

On drawing out the example for himself the reader will find that even if the mechanism be drawn full size great care is necessary to obtain anything like an accurate result.

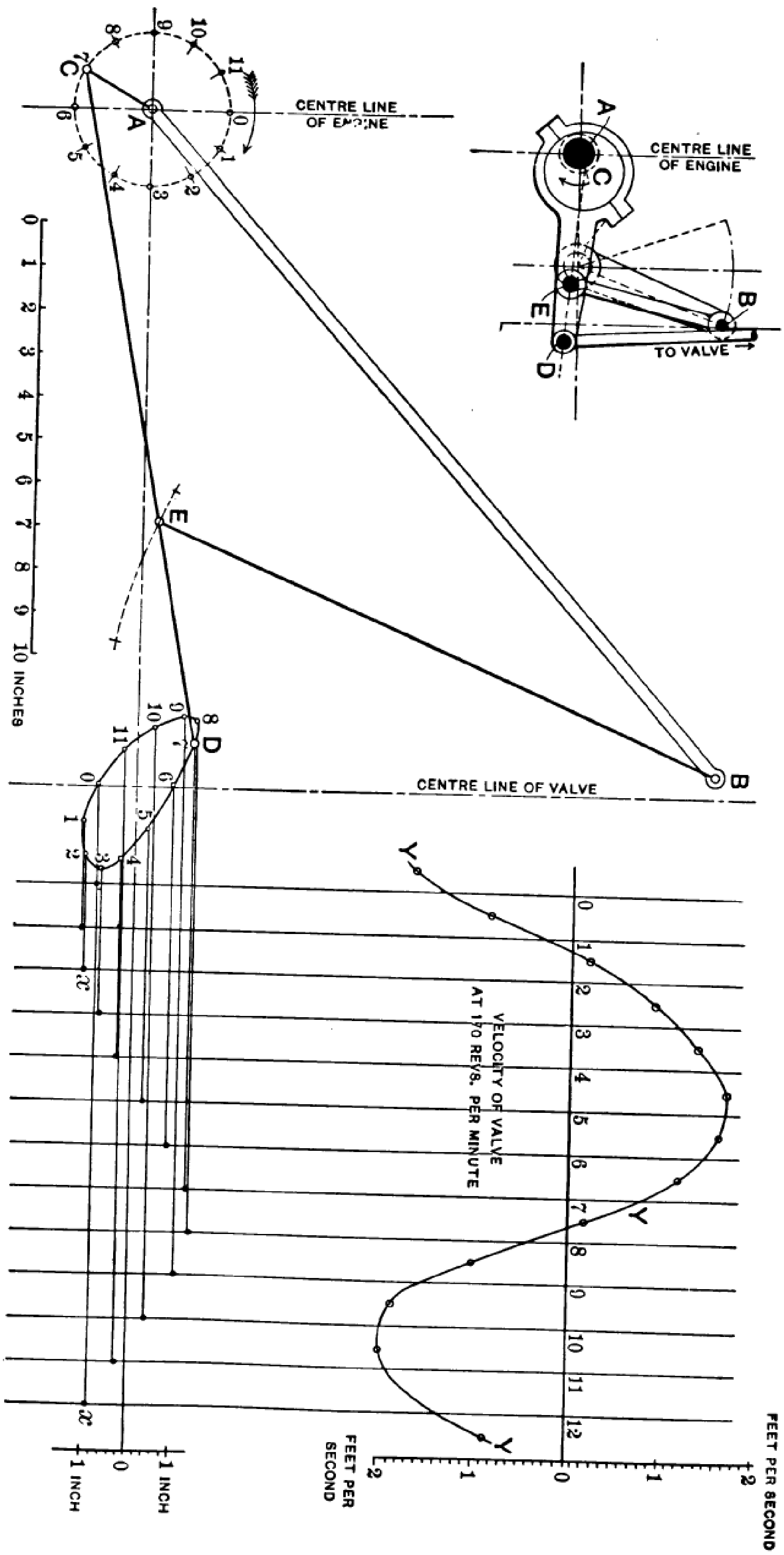
CENTRE LINE OF ENGINE
TO VALVE
CENTRE LINE OF VALVE
10 INCHES
VELOCITY OF VALVE AT 170 REVS. PER MINUTE
FEET PER SECOND
1 INCH

FIG. 97.

50. Polar Diagrams of Velocities for Simple Plane Mechanisms.—The velocities in plane mechanisms can only be determined graphically from the positions of the virtual centres of the links when these centres fall within the limits of the drawing, and when their positions can be found with accuracy. Often the exact position of a virtual centre is difficult to define, because it lies at the intersection of two lines which make a very small angle with one another.

To avoid these difficulties, a general method of drawing diagrams for velocities and accelerations of points in mechanisms has been devised,* and a few simple cases will be considered here.

In Fig. 98 let ABC represent a rigid body having plane motion, and suppose the linear velocity v of the point A and the angular velocity ω of the whole body to be known. It is required to determine the linear velocities of the points B and C.

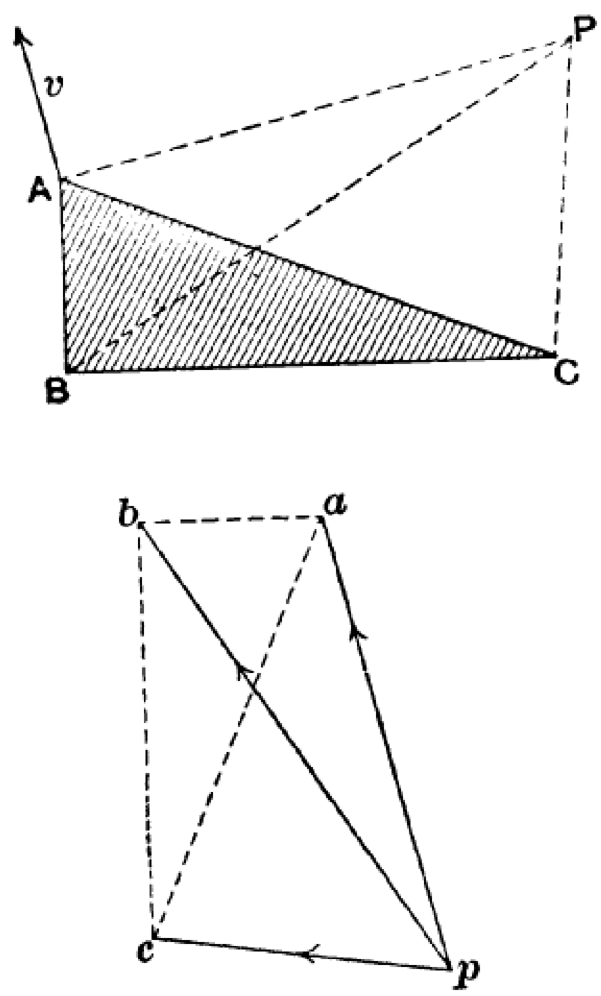

FIG. 98.

Let P be the virtual centre of ABC with regard to the plane of motion; then $AP=\frac{v}{\omega}$; hence the position of P can be found, since PA is perpendicular to the direction of v.

Join PB, PC. From these lines the directions of motion of B and C are known, and the magnitudes of the linear velocities are also known, since

$$\text{velocity of } B=\omega\times PB,$$

and

$$\text{velocity of } C=\omega\times PC.$$

* R. H. Smith, Graphics, Book I, Chap. IX ; Burmester, Kinematik, Chaps. XI and XII.

These velocities can, however, be determined (without finding the position of the virtual centre) as follows:

From any pole p draw the vector pa, representing the velocity v. From the point a draw $ab(=\omega \times AB)$ perpendicular to AB, and draw bc $(=\omega \times BC)$ perpendicular to BC. Then pb, pc are vectors representing respectively the linear velocities of the points B and C. The truth of this statement will be seen from the facts that the sides pa, ab are perpendicular to the sides PA, AB, and they are also proportional, since

$$pa = \omega \times PA,$$

and $$ab = \omega \times AB.$$

Hence $$pb = \omega \times PB = \text{velocity of } B.$$

And similarly $$pc = \omega \times PC = \text{velocity of } C.$$

Note also that the triangles abc, ABC are similar; in fact abc is the *velocity image* of the body ABC, and is turned through an angle of 90° *in the same sense as that of the angular velocity* ω. The lines ab, bc, ca are of course vectors, and on consideration it will be evident that ab, for instance, represents the linear velocity of B (round A as centre), due to the actual angular velocity ω, because we have drawn $ab = \omega . AB$ and at right angles to AB. Further, the values of pb and pc have been obtained by vector-addition, the process described and explained in § 16.

Next suppose that we have to determine the velocities in a linkwork mechanism such as that of Fig. 99, where $ZPABC$ represents a chain of links connected by turning pairs, PZ being the fixed link.

The angular velocity ω of PA is supposed to be known, and also the directions in which the points B and C are moving at the instant considered.

The point A is turning about the fixed point P; hence its direction of motion is at right angles to PA and its linear velocity is $\omega . PA = v_A$. We have to find v_B and v_C, the linear velocities of B and C.

Take any pole p, and draw lines pa, pb, pc from it par-

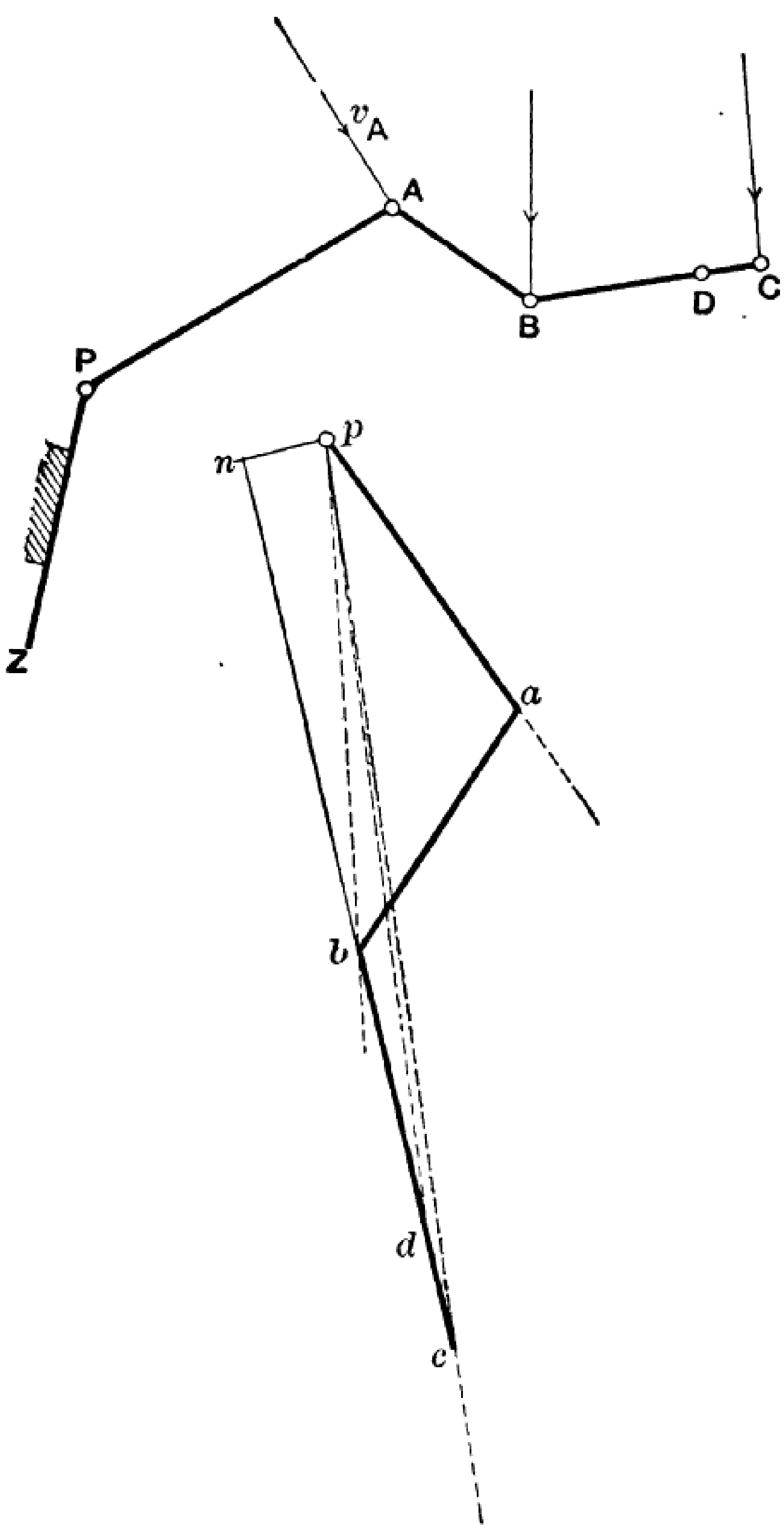

FIG. 99.

allel to the given directions of v_A, v_B, and v_C. Set off $pa = v_A$ and draw ab perpendicular to AB and bc perpendicular to

BC. Then *pb* and *pc* represent v_B and v_C on the same scale as that on which *pa* represents v_A.

The triangle *pab*, for example, is a vector triangle, or triangle of velocities, for the body *AB*, and *ab* is the velocity image of *AB*, just as in the previous case. The vector *ab* really represents the velocity of *A* with regard to *B* or of *B* with regard to *A*, according to the sense in which we measure it.

It is evident that the linear velocity of the point *B* will be due to two causes: (1) the velocity of *A* with regard to the fixed link *ZP*, and (2) that of *B* with regard to *A*. We also know that *B* can have no velocity along *BA*, for *BA* is a rigid body.

Hence to find *pb* (the velocity of the point *B* with regard to *ZP*) we compound *pa* (the velocity of *A* with regard to *ZP*) with *ab* (the velocity of *B* with regard to *A*). Similar reasoning holds good in the case of *pc*.

If *cb* be produced and a line, *pn*, drawn to cut it at right angles, it will be seen that *pc* is the resultant of *pn* (the velocity of *C* in the direction *CB*) and *nc* (the velocity of *C* in the direction normal to *CB*). Similarly *nb* is the velocity of *B* in the direction normal to *CB*, and hence *bc* is the velocity of *C with regard to B*, measured in a direction normal to *CB*, i.e., *bc* is the velocity of *C* about *B*. Note that since the link *CB* is rigid, *C* and *B* must have the same velocity, *pn*, along *CB*.

The diagram can be drawn in a similar way if some of the pairs are sliding pairs, and it will be found that if the chain of links has both ends attached to fixed points the velocity diagram becomes a closed polygon.

If it is required to find the velocity of a point *D* in the link *BC*, that of *C* being known, it is only necessary to make $bd = bc \times \frac{BD}{BC}$, and to join *pd*. The required velocity is then represented by *pd*. This is evident, since the veloc

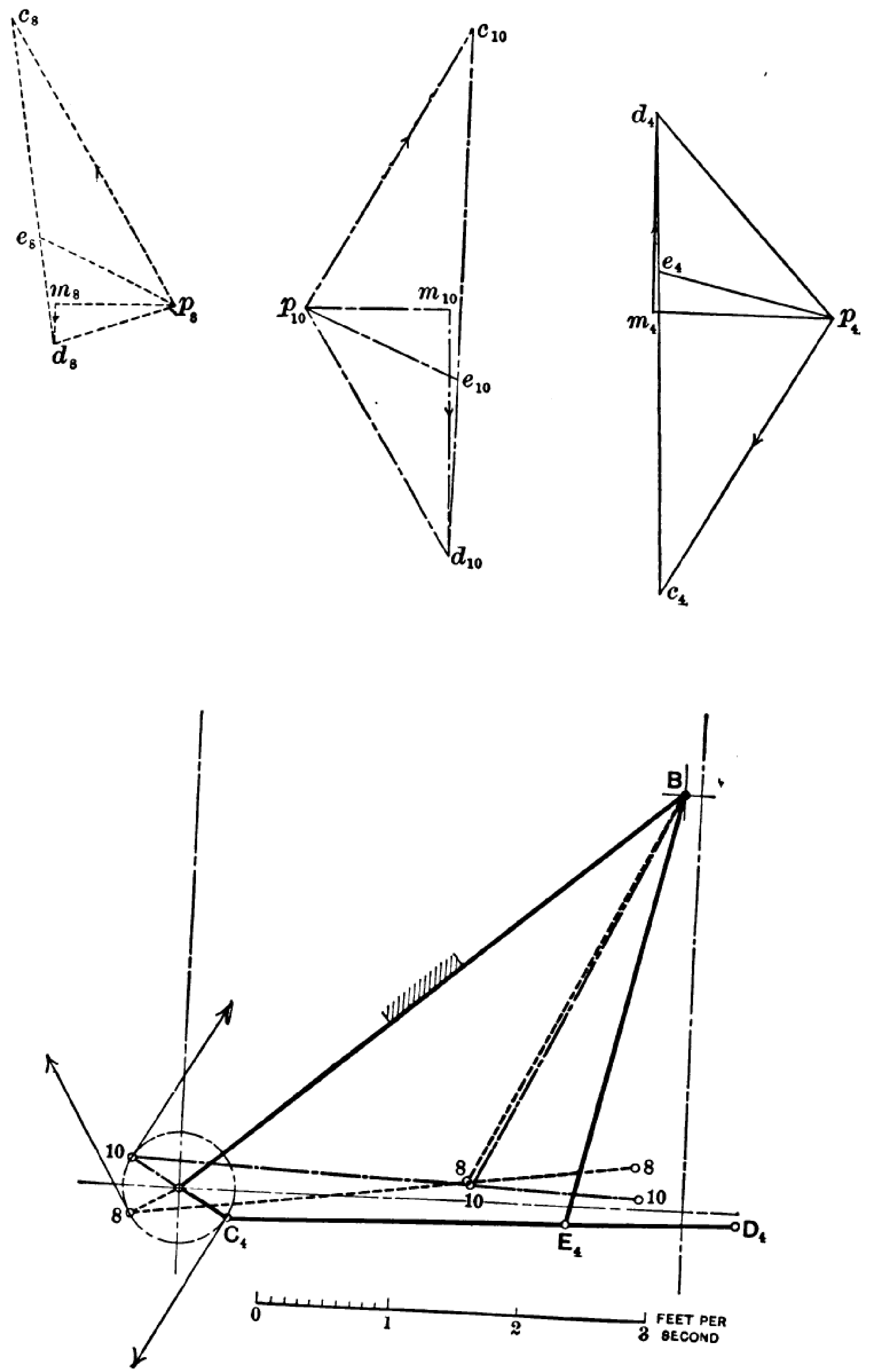
c_8
e_8
m_8
p_8
d_8
c_{10}
p_{10}
m_{10}
e_{10}
d_{10}
d_4
e_4
m_4
p_4
c_4
B
10
8
8
10
8
10
C_4
E_4
D_4
0
1
2
3
FEET PER SECOND

FIG. 100.

ity of D about B is to that of C in the proportion of $BD : BC$, hence $\frac{bd}{bc} = \frac{BD}{BC}$. Thus bcd is seen to be the velocity image of BCD.

We may take as an example of the use of this construction the Bremme valve-gear of Fig. 97. In Fig. 100 the velocity diagrams have been drawn for the positions 4, 8, and 10 of Fig. 97. The diagrams have been drawn from separate poles for the sake of clearness, but they might equally well have been drawn from the same point as pole if that had been advisable.

Having drawn the mechanism in position 4 (say), and having found by calculation that v_c (the velocity of the centre of the eccentric) is 2.59 feet per second, a line p_4c_4 is drawn from the pole in a direction parallel to v_c, and of the proper length. We know that the direction in which E is moving is perpendicular to BE, and p_4e_4 is therefore drawn of indefinite length perpendicular to BE. The point e_4 is found by drawing c_4e_4 perpendicular to C_4E_4 and p_4e_4 then gives the magnitude of the velocity of E. To find that of D, we produce c_4e_4 to d_4, making $\frac{cd}{ce} = \frac{CD}{CE}$; then p_4d_4 represents the velocity of D when the mechanism is in position 4. The vertical velocity of the valve will be represented by d_4m_4, the vertical velocity of the point D to which it is attached, and on measurement this line is found to scale 1.56 feet per second. (Compare value shown by curve in Fig. 97.) In a similar way d_8m_8 and $d_{10}m_{10}$ are found, and so on for any required position of the gear.

It is not difficult to see that the velocity diagrams obtained by this method are really the same as some of those whose construction in certain special cases is explained in Chapters III and IV. For instance, Fig. 101 shows the construction already described for the piston velocity in a direct-acting engine, together with the polar method of determining the same quantity. It is plain that the triangles BCE and bpa are similar and that pa and CE represent the same quantity to different scales. The vector pa thus represents the linear velocity of the piston, while pb

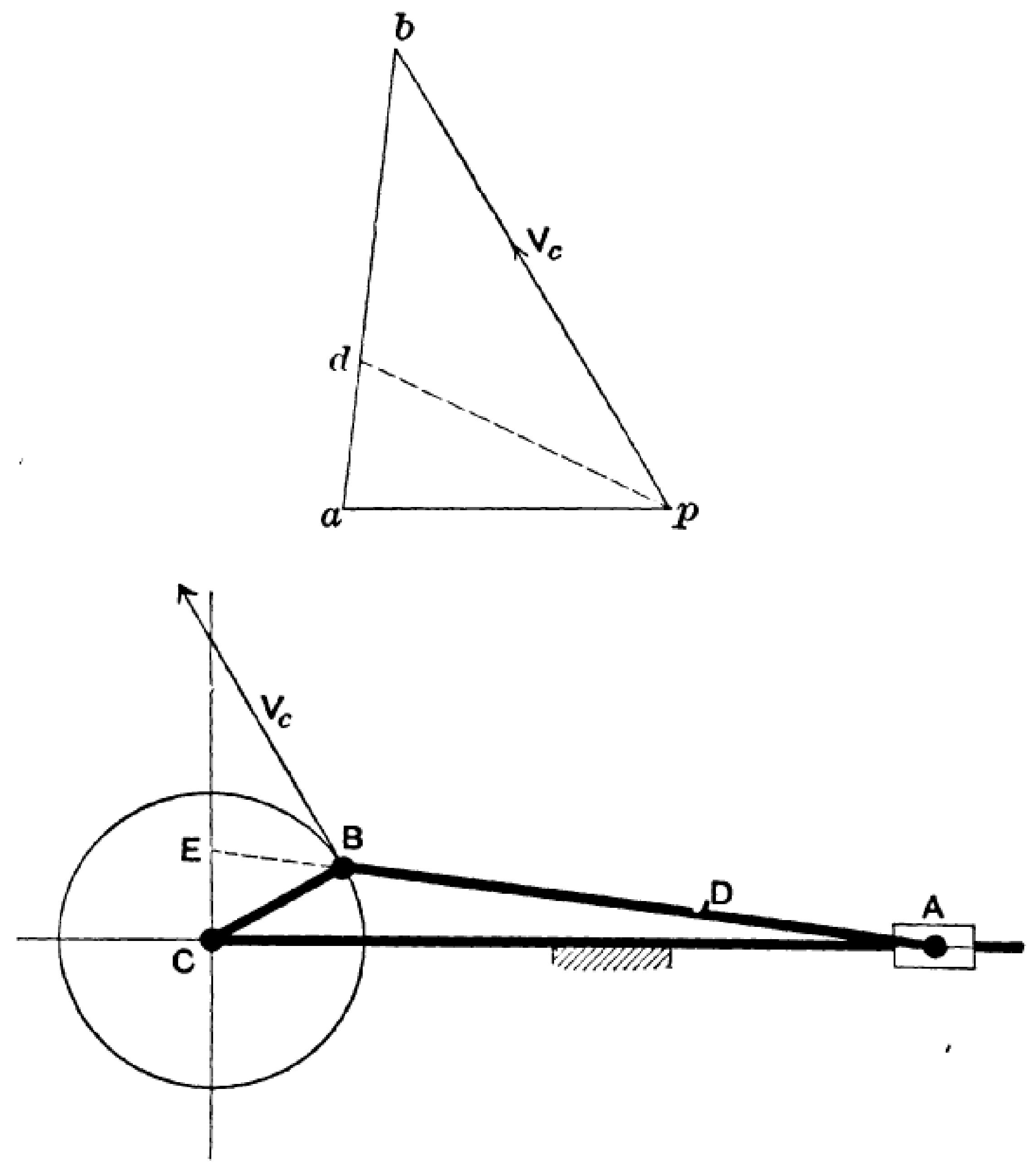

FIG. 101.

and ab represent respectively the linear velocity of the crank-pin B around C and that of the crank-pin B around A. ab is in fact the *velocity image* of AB, and pd gives the velocity of any point D on AB, if $\frac{ad}{ab} = \frac{AD}{AB}$. Since the triangles BCE and bpa are similar it also follows that we may look on BE as a velocity image of BA. It is, however, turned through an angle of 90° from the position ab.

51. Indirect Method in more Complex Cases.—It is not possible in every case to proceed in such a direct manner in

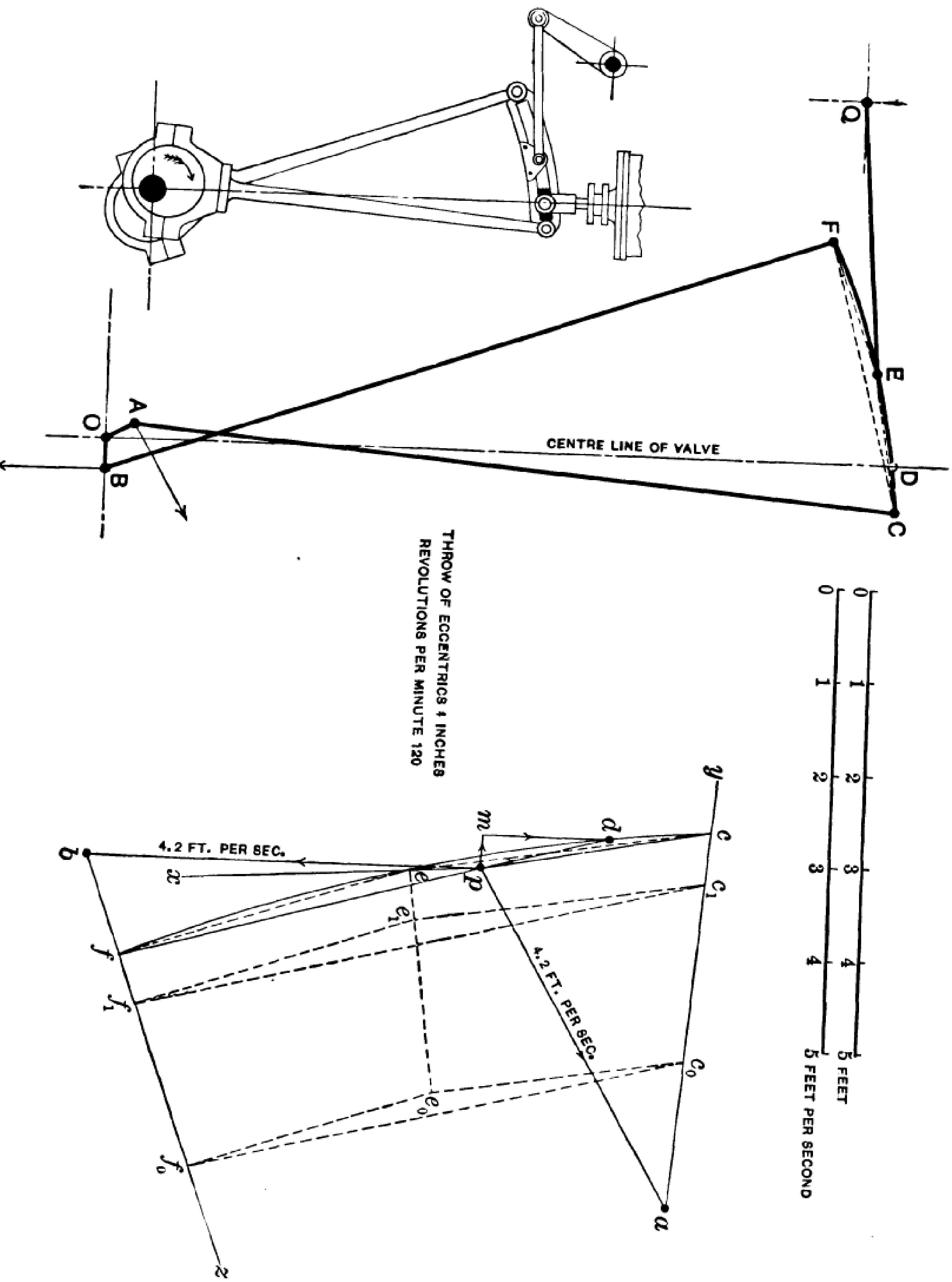

FIG. 102.

constructing the velocity diagram for a mechanism. Fig. 102 shows a link motion for working the slide-valve of a steam-engine, of which OA, OB are the two eccentrics. These are rigidly connected and rotate uniformly about O, the centre of the crank-shaft; AC, BF are the eccentric-rods, CF the link, and QE is the drag-rod or suspension-link. Q remains fixed, except when the engine is reversed. We wish to find the velocity of a point D on the link CF. The motion for the valve is taken from the point D.

The actual velocities of points A and B are known, and also the direction of motion of E.

Having drawn out the mechanism in the required position, a pole, p, is taken and the vectors pa, pb drawn representing the linear velocities of A and B respectively. In the figure these correspond to an angular velocity of 120 revolutions per minute; they are each 4.2 feet per second.

A line px of indefinite length is next drawn at right angles to QE, and therefore parallel to the direction of motion of E. The point e must of course lie somewhere on this line. We next draw ay, bz of indefinite length, perpendicular to AC and BF respectively; the points c and f must lie somewhere on these lines.

The required velocity images of the points C, E, and F must lie in some such position as $c_0e_0f_0$, where the triangle $c_0e_0f_0$ is similar to the triangle CEF, but is rotated through 90° in the sense of the motion. Another possible position would be $c_1e_1f_1$, and the line e_0e_1 will evidently pass through all possible positions of e. Thus e is found at the intersection of the lines e_0e_1 and px. We then draw ec and ef respectively perpendicular to EC and EF, and through the points c e and f draw a circular arc, which will be the velocity image of the curved link CEF.

The vectors pc, pe, pf then give the velocities of C, E, and F respectively. To find the velocity of D the point d is marked in its proper position on the curve cef, and pd is drawn. In the position shown this velocity is 1.42 feet per second, and is not in the direction of the centre line of the