CHAPTER IV.

SLIDER-CRANK CHAINS.

34. **Slider-crank Chain.**—A very important chain is obtained from the quadric crank-chain by substituting a sliding pair for one of the turning pairs. It is obvious that the links will undergo the same relative change of position in Fig. 60 (*b*) as in Fig. 60 (*a*), although the lever *c* has been replaced by a block sliding in a circularly curved slot of the same radius as the original lever. The chain as thus transformed may be called a cylindric slider-crank chain, although this name is generally applied to the particular case in which O_{cd} is at an infinite distance and the block slides in a straight slot. It is plain that the mechanism of Fig. 60 (*c*) may be obtained from that of Fig. 60 (*b*) by continually increasing the radius of the pair *cd* until it becomes infinite. The pair *cd* may have prismatic surfaces of any form so long as the sliding motion is properly constrained; thus, for example, *c* may be a hollow block sliding on a prismatic rod *d*, Fig. 60 (*c*). The slider-crank chain in its cylindric form has of course plane motion, and is of special importance, since its different inversions form amongst others the mechanisms of various types of reciprocating steam-engines.

The six virtual centres of the slider-crank chain are easily found, exactly as in the case of the quadric crank-chain, but O_{cd} is always inaccessible. Fig. 61 shows the centrodes of the links *b* (representing the connecting-rod of a direct-acting engine) and *d* (representing the frame or bedplate). The centrode of *b* with respect to *d* (i.e., if *d* is considered as the fixed link) is shown by the full line; the dotted curve represents the centrode described by O_{bd} if b

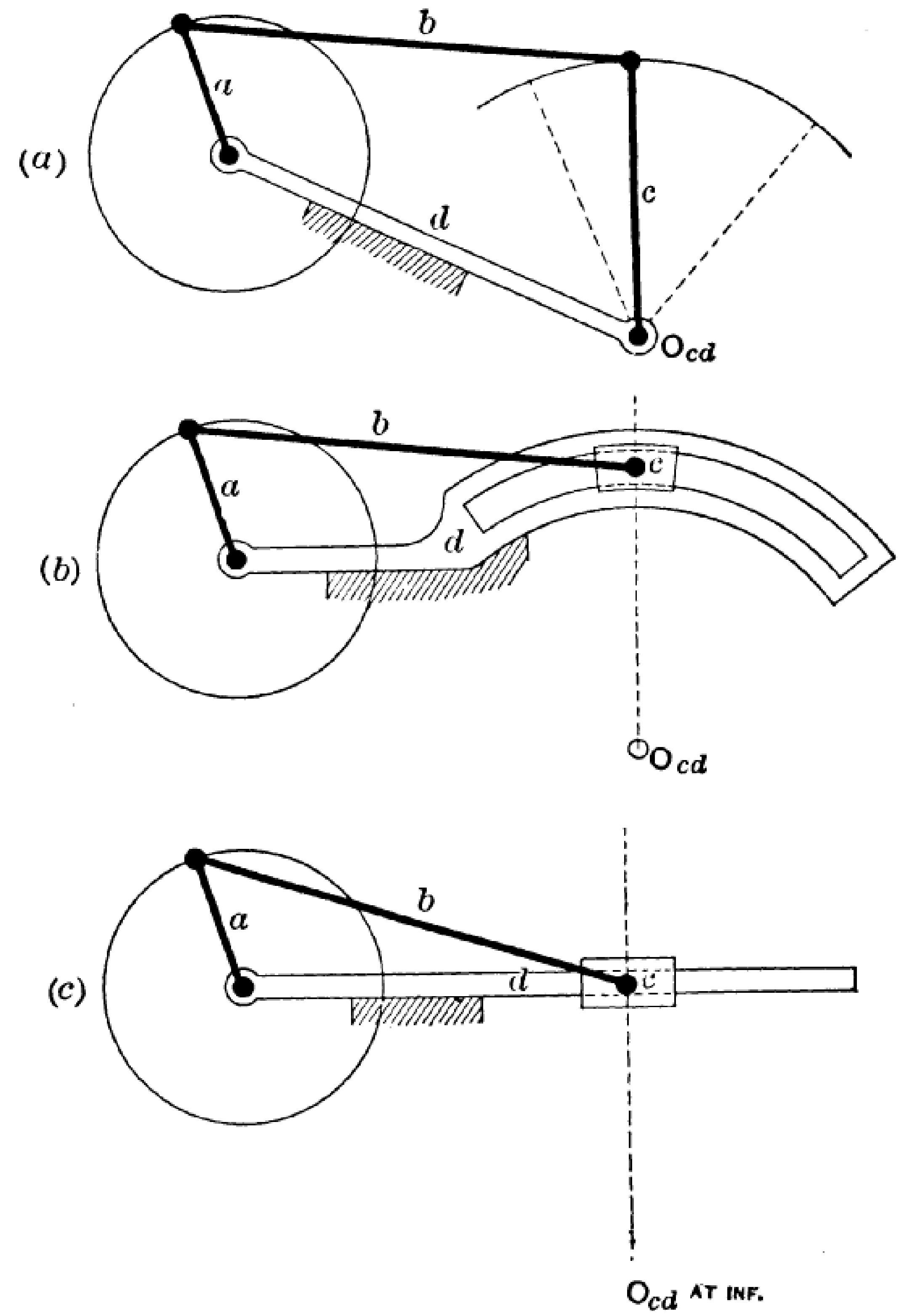

FIG. 60.

is taken as the fixed link. The construction for one point is shown in each case.

When d is fixed the link c represents the piston, piston-rod, and cross-head of the same machine. The link a represents the crank, and b the connecting-rod. A point on the link b between A and B describes an oval curve with reference to d, the shape depending on the position of the point selected, and on the ratio of the lengths of crank and con-

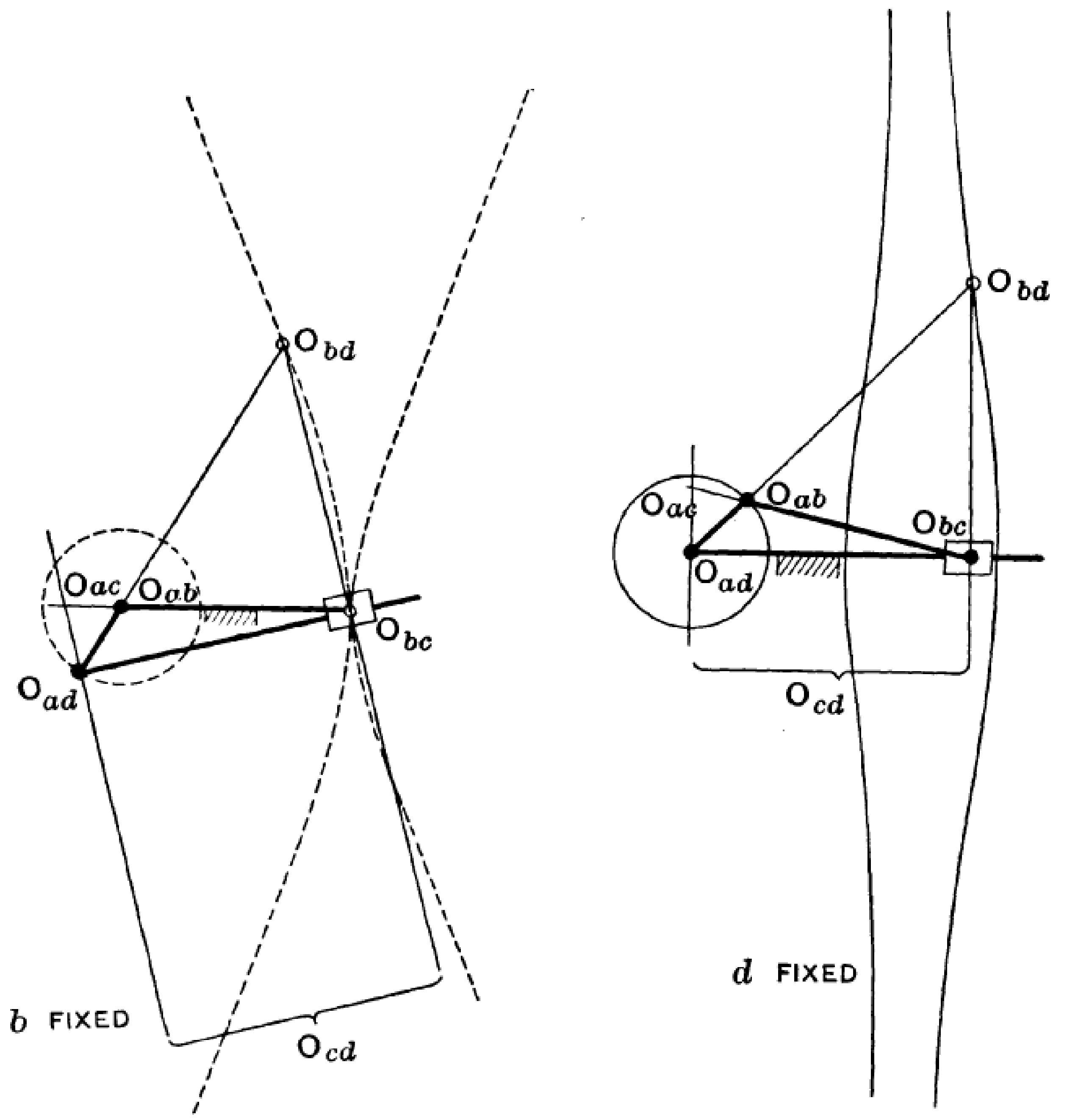

FIG. 61.

necting-rod. This fact is utilized in the design of certain valve-gears.

35. Displacement, Velocity, and Acceleration of Cross-head in Direct-acting Engine. (First Inversion of Slider-crank Chain.)—One of the most important problems in connection with the slider-crank chain is the determination of the velocity and acceleration of the link c, Fig. 60, supposing d to be fixed, and a to rotate with uniform angular velocity. This is approximately the case in a direct-acting steam-engine, where c would represent the cross-head and b the connecting-rod.

It is in general most convenient to deal with these problems graphically, but we shall first give an analytical investigation.

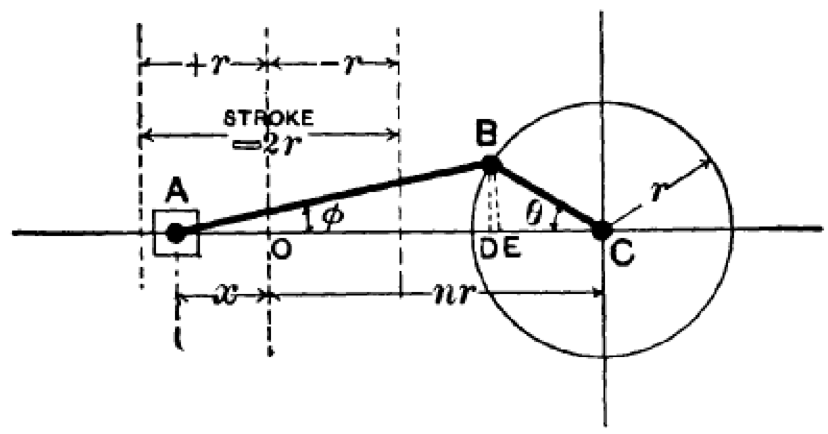

FIG. 62.

In Fig. 62, suppose the line of stroke AO to pass through C, the centre of the crank-shaft. Let BC (the throw of the crank) $=r$, and let $\frac{AB}{BC}=n$, so that the length of connecting-rod $=nr=AB$. When the crank makes any angle θ with the centre line AC, let x be the distance of the cross-head A from O, the middle of its stroke. Draw BD perpendicular to AC, and mark off $AE=AB$. If φ is the angle of obliquity of the connecting-rod,

$$\sin\varphi=\frac{\sin\theta}{n},\quad\text{and}$$

$$\cos\varphi=\frac{1}{n}\sqrt{n^2-\sin^2\theta}.$$

Now
$$\begin{aligned}x&=AC-OC=AC-AB\\&=CD+DA-AB\\&=r\cos\theta+nr\cos\varphi-nr\\&=r\left(\cos\theta-n+\sqrt{n^2-\sin^2\theta}\right).\qquad(1)\end{aligned}$$

This gives x in terms of the crank angle θ. It is to be noticed that when $\theta=\frac{\pi}{2}$ the cross-head is not at the middle of its stroke, but at a distance

$$\begin{aligned}x_0&=r\left(\sqrt{n^2-1}-n\right)\\&=-\frac{r}{\sqrt{n^2-1}+n},\end{aligned}$$

the negative sign indicating that A is now to the right of O, Fig. 62.

In the case of a cross-head having simple harmonic motion we should have simply

$$x = r \cos \theta.$$

The term $r(\sqrt{n^2 - \sin^2 \theta} - n)$ in equation (1) thus gives what is called the "error due to obliquity" of the connecting-rod. Its values for $\theta = \frac{\pi}{-}$ are shown below for some usual values of n.

$n =$	4	5	6
$\sqrt{n^2 - \sin^2 \theta} - n =$	-0.13	-0.11	-0.09

The error due to obliquity is thus seen to diminish rapidly as n increases.*

Next, to determine the velocity of the piston at any instant we differentiate x with regard to time and obtain

$$\frac{dx}{dt} = r\left[-\sin \theta \frac{d\theta}{dt} + \frac{1}{2}(n^2 - \sin^2 \theta)^{-\frac{1}{2}} \frac{d}{dt}(n^2 - \sin^2 \theta)\right]$$

$$= -r\frac{d\theta}{dt}\left[\sin \theta + \frac{2 \sin \theta \cos \theta}{2\sqrt{n^2 - \sin^2 \theta}}\right].$$

This is not very convenient for use in practice, but for ordinary values of n we may write without large error n instead of $\sqrt{n^2 - \sin^2 \theta}$. For example, if $\theta = \frac{\pi}{2}$, and $\sin \theta$ has its greatest value,

	$\sqrt{n^2 - \sin^2 \theta} = 3.87$	4.89	5.91
when	$n = 4$	5	6

Further, we may write V_c, the linear velocity of the crank-

* For a discussion of the problem of the connecting-rod see Hill, Min. Proc. Inst. C. E., Vol. CXXIV, p. 390. Also consult Unwin, Min. Proc. Inst. C. E., CXXV, p. 363, and a paper by G. A. Burls, Min. Proc. Inst. C. E., Vol. CXXXI, p. 338.

pin, instead of $r\frac{d\theta}{dt}$, and, omitting the negative sign, which simply shows that x diminishes at first while θ increases, we have very approximately for the velocity of the piston or cross-head

$$V_p = V_c\left(\sin\theta + \frac{\sin 2\theta}{2n}\right). \quad . \quad . \quad . \quad . \quad (2)$$

As an example, suppose an engine 12 inches stroke running at 250 revolutions per minute, the length of connecting-rod being 3 feet. The crank-pin velocity will be $\frac{250 \times 3.14}{60} =$ 13.08 feet per second. When $\theta = 45°$, the value of n being 6, we have, from equation (2),

$$\begin{aligned} V_p &= 13.08(0.70711 + 0.08333) \\ &= 13.08 \times 0.79044 \\ &= 10.340 \text{ feet per second.} \end{aligned}$$

If the velocity were calculated from the accurate expression previously obtained, we should get

$$\begin{aligned} V_p &= 13.08\left(0.70711 + \frac{1}{2\sqrt{36 - 0.4998}}\right) \\ &= 13.08 \times 0.79103 \\ &= 10.348 \text{ feet per second.} \end{aligned}$$

The approximation, therefore, has led to an error of only 0.008 foot per second in this case.

Proceeding to determine the acceleration of the piston or cross-head for any crank angle, we find very approximately from equation (2), remembering that V_c is constant,

$$\frac{d}{dt}(V_p) = V_c\left(\cos\theta\,\frac{d\theta}{dt} + \frac{1}{2n}\left[2\cos 2\theta\frac{d\theta}{dt}\right]\right)$$

Now $\frac{d\theta}{dt} = \frac{V_c}{r}$; thus

$$\text{acceleration of piston or cross-head} = \frac{V_c^2}{r}\left(\cos\theta + \frac{\cos 2\theta}{n}\right). \quad (3)$$

The following table gives the value of $\cos\theta + \cos\frac{2\theta}{n}$ for different values of θ and n.

θ.	Value of n.					
	4	4.5	5	5.5	6	∞
0° or 360°.......	1.250	1.222	1.200	1.182	1.167	1.000
30° or 330°.......	0.991	0.977	0.966	0.957	0.949	0.866
60° or 300°.......	0.375	0.389	0.400	0.409	0.417	0.500
90° or 270°.......	−0.250	−0.222	−0.200	−0.182	−0.167	0.000
120° or 240°.......	−0.625	−0.611	−0.600	−0.591	−0.583	−0.500
150° or 210°.......	−0.741	−0.755	−0.766	−0.775	−0.783	−0.866
180°.......	−0.750	−0.778	−0.800	−0.818	−0.833	−1.000
	Values of $\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$					

36. Graphic Methods for Cross-head Velocity and Acceleration.—We proceed to consider graphic means of determining velocity and acceleration for the cross-head or piston of a direct-acting engine. It is of course possible to draw first a curve of displacement on a time base, and then use the methods described in Chapter II, but simpler means can be employed in this case. In Fig. 63 let AB, BC represent

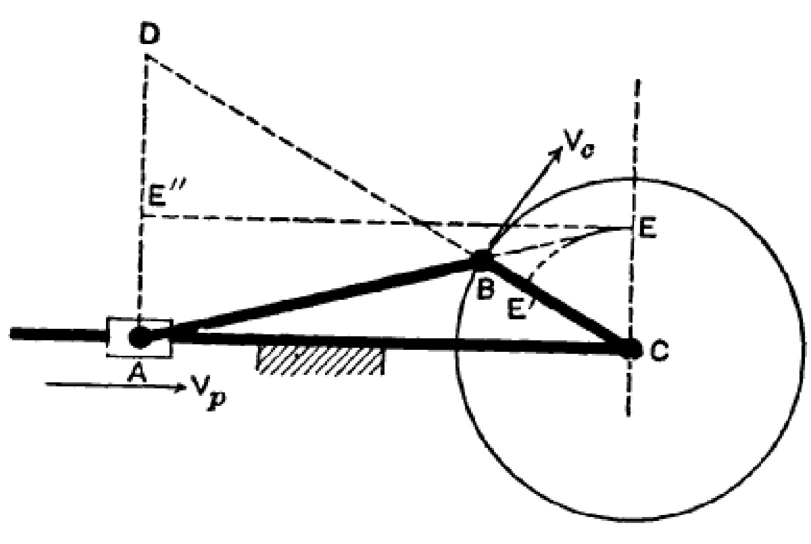

FIG. 63.

sent the connecting-rod and crank in any given position. The point A is moving along the straight line AC, while B is moving for the instant in a direction perpendicular to BC. Hence D, the virtual centre of AB with regard to the fixed link, is easily found at the intersection of the virtual radii of the points A and B. Through C draw a line perpendicular to AC, and therefore parallel to AD, and produce AB to meet it in E. Then the triangles ADB, ECB are similar.

Now the angular velocity of AB about D is measured either by the ratio $\frac{V_p}{AD}$ or by $\frac{V_c}{BD}$, so that

$$\frac{V_p}{V_c} = \frac{AD}{BD} = \frac{CE}{CB}.$$

In many positions of the mechanism D is inaccessible, but E can always be found, and the relation just obtained tells us that CE *represents the velocity of the piston* at the instant for which the diagram is drawn, to the same scale as that to which CB represents the velocity of the crank-pin.

It is generally most convenient to make a polar diagram of piston velocity by marking off a series of points such as E' (where $CE' = CE$) for a number of different crank positions, or, if required, a velocity diagram on a distance base may be constructed by marking off the distance CE along AD, so that a series of points such as E'' are obtained, and a curve drawn whose ordinate at any point is proportional to the velocity of the piston when in that position. Such diagrams have been drawn in Fig. 64, together with a linear velocity diagram on a time base, so as to show the difference between a simple harmonic motion and that which the piston actually possesses. The example taken is that for which the velocity and acceleration have been calculated in the last section. In order to determine the scale to which the ordinates of the curves represent the velocity, it is only necessary to remember that if the length BC were 1 inch, the velocity scale would be 1 inch = 13.08 feet per second, since the crank-pin velocity is 13.08 feet per second. In the figure the construction lines are shown for one position of the mechanism only; in drawing such diagrams care should be taken only to draw those portions of the construction lines which are absolutely necessary, so as to avoid useless complication. Of course accuracy in drawing is indispensable if the numerical results obtained are to be reliable. A line whose length is proportional to the *piston acceleration*

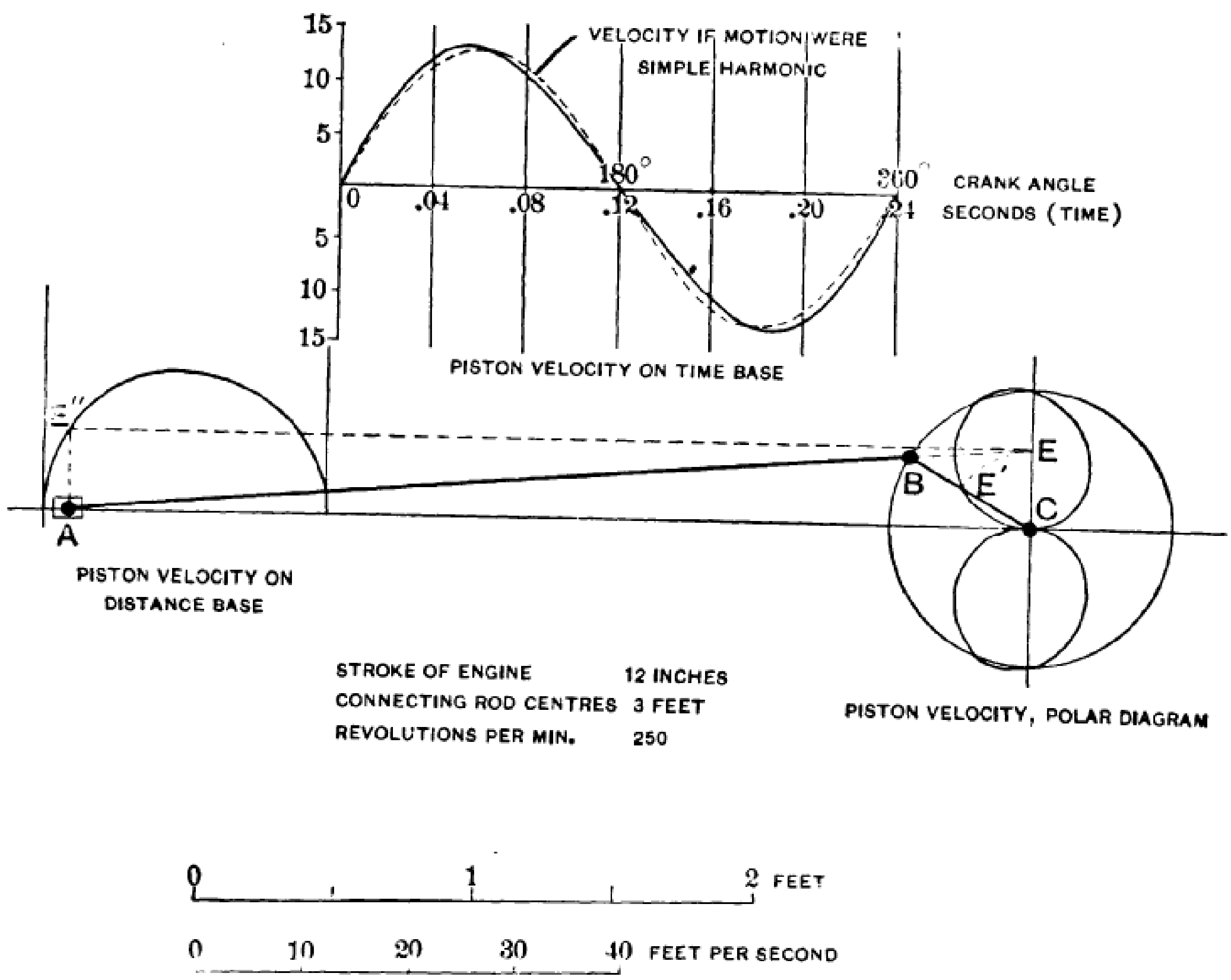

FIG. 64.

may be obtained as follows (see Fig. 65): Take any given position of the cross-head A, and produce AB to cut CE in E. Then, as before,

$$\frac{CE}{CB} = \frac{V_p}{V_c}.$$

Notice that this is true whether the path of the point A passes through C or not, when produced.

The desired acceleration is the rate of change of V_p, which is of course proportional to the rate at which the distance CE is increasing or diminishing at the instant considered. In fact the piston *acceleration* may be considered as being proportional to the *velocity* of the point E along CE at any instant while the engine is in motion, supposing BE always to be in a straight line with AB.

Let this velocity along CE be u_0. The real velocity of the point E, regarded as a point on the connecting-rod, is in a direction perpendicular to DE, its virtual radius. Calling this velocity u_1, we see that u_1 may be resolved into two

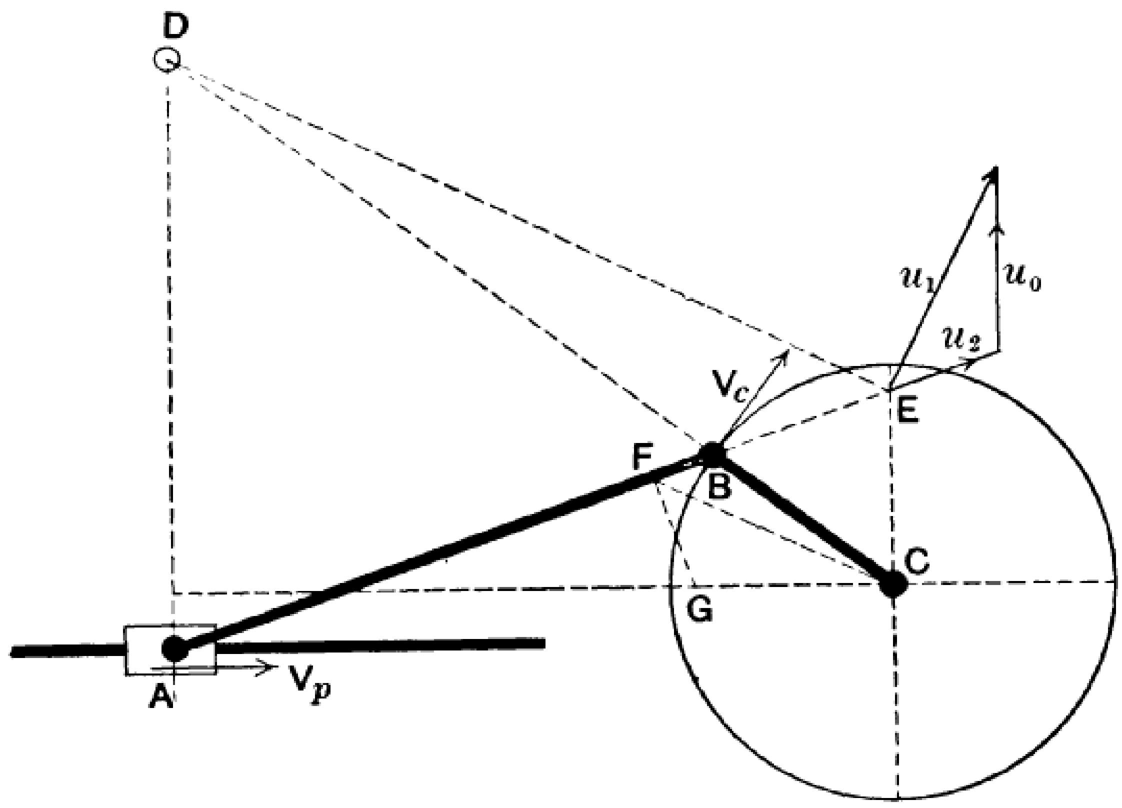

FIG. 65.

components, namely, u_0 in a direction along CE, and u_2 in a direction along BE.

From C draw CF parallel to DE, and draw FG perpendicular to AB. Then the sides FC, CG, GF, of the triangle FCG are respectively perpendicular to the directions of u_1, u_0, u_2. Thus FCG is a triangle of velocities and $\frac{u^0}{u_1}=\frac{CG}{FC}$, or $u_0=u_1\frac{CG}{FC}$. But $\frac{u_1}{V_c}=\frac{DE}{DB}=\frac{FC}{CB}$, therefore $u_1=V_c\frac{FC}{CB}$, and

$$u_0=V_c\frac{CG}{CB}=\text{rate of change of length } CE.$$

Now it has been shown that

$$\text{piston velocity}=V_p=CE.\frac{V_c}{CB},$$

and V_c and CB are constant; hence it follows that the rate of change of the piston velocity must be equal to

$$(\text{rate of change of } CE)\times\frac{V_c}{CB}, \text{ that is,}$$

$$\text{piston acceleration} = u_0\frac{V_c}{CB} = \frac{V_c^2}{CB^2}.CG.$$

Thus to obtain the numerical value of the piston acceleration we must multiply the length of CG (measured to scale in feet) by $\left(\frac{V_c}{r}\right)^2$, where V_c is the crank-pin velocity in feet per second and r is the crank throw, or radius of the crank-pin circle, in feet.

Hence it follows that

$$\frac{CG}{CB} = \frac{\text{acceleration of piston}}{V_c^2/r},$$

or, in other words, CG represents the piston acceleration to the same scale as that on which CB represents V_c^2/r, the radial acceleration of the crank-pin.

When drawing such a diagram as Fig. 65 it happens that for many positions of the crank the point D becomes inaccessible. Accordingly some other construction must be found to obtain the position of the point F, so that CG may be determined for any crank angle.

Consider the triangles BEC and BAD.

Evidently

$$\frac{BE}{BA} = \frac{BC}{BD}.$$

But $\frac{BC}{BD} = \frac{BF}{BE}$, because the triangles BDE, BCF are similar.

Therefore

$$\frac{BE}{BA} = \frac{BF}{BE},$$

or

$$BA\,.\,BF = BE^2.$$

Hence any construction which will make BE a mean proportional between BA and BF will determine the point F.

A number of such constructions have been given; of these perhaps the most convenient in practice is that of Kisch.*

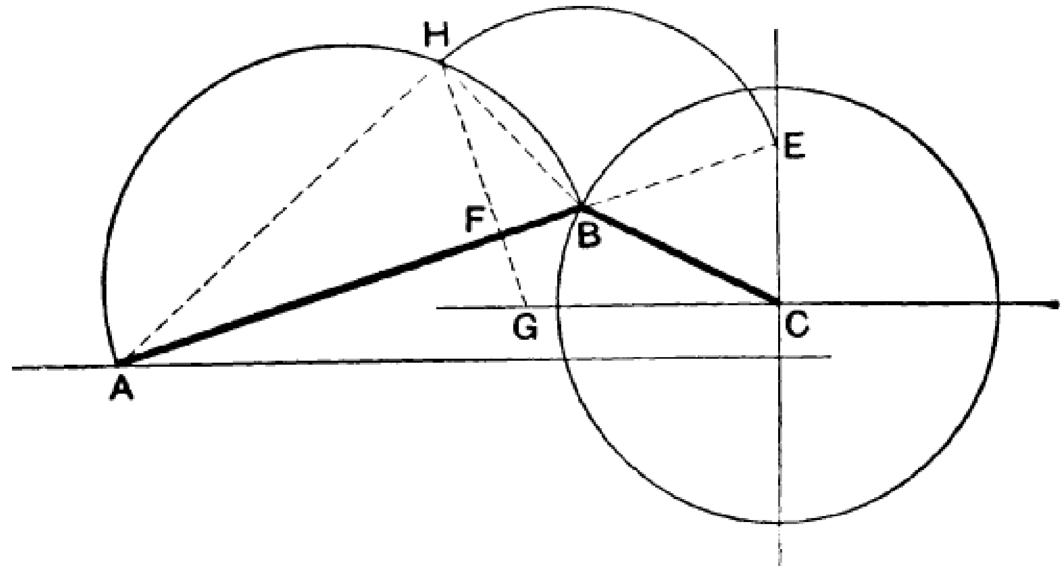

FIG. 66.

On AB describe a semicircle AHB. With centre B and radius BE cut the semicircle in H. Draw HFG perpendicular to AB, cutting AB in F and CG in G. Join BH, HA. Then

$$\frac{BF}{BH}=\frac{BH}{BA}.$$

But $BE=BH$. Hence $BA.BF=BE^2$, and CG represents the acceleration.

The method of determining the acceleration scale of such a diagram may be shown by a numerical example. Fig. 67 has been drawn for the engine for which the velocity of the piston has been previously calculated, taking a crank angle of 45°. The crank-pin velocity being 13.08 feet per second, and the connecting-rod being 6 cranks in length, we have for the acceleration of the piston at that particular crank angle

$$\text{acceleration}=\frac{13.08^2}{0.5}\left(\cos 45^\circ+\frac{\cos 90^\circ}{6}\right)$$
$$=\frac{13.08\times 13.08\times 0.70711}{0.5}$$
$$=242.1 \text{ feet per second per second.}$$

* See Zeitschrift des Vereines Deutscher Ingenieure, Dec. 13, 1890. Given also by Klein, Journal of Franklin Inst., Vol. CXXXII, Sept. 1891.

In Fig. 67 the actual length of the line CG, if the figure were drawn the full size of the engine, would be 0.351 foot. The radius of the crank-pin circle CB is 0.5 foot and repre-

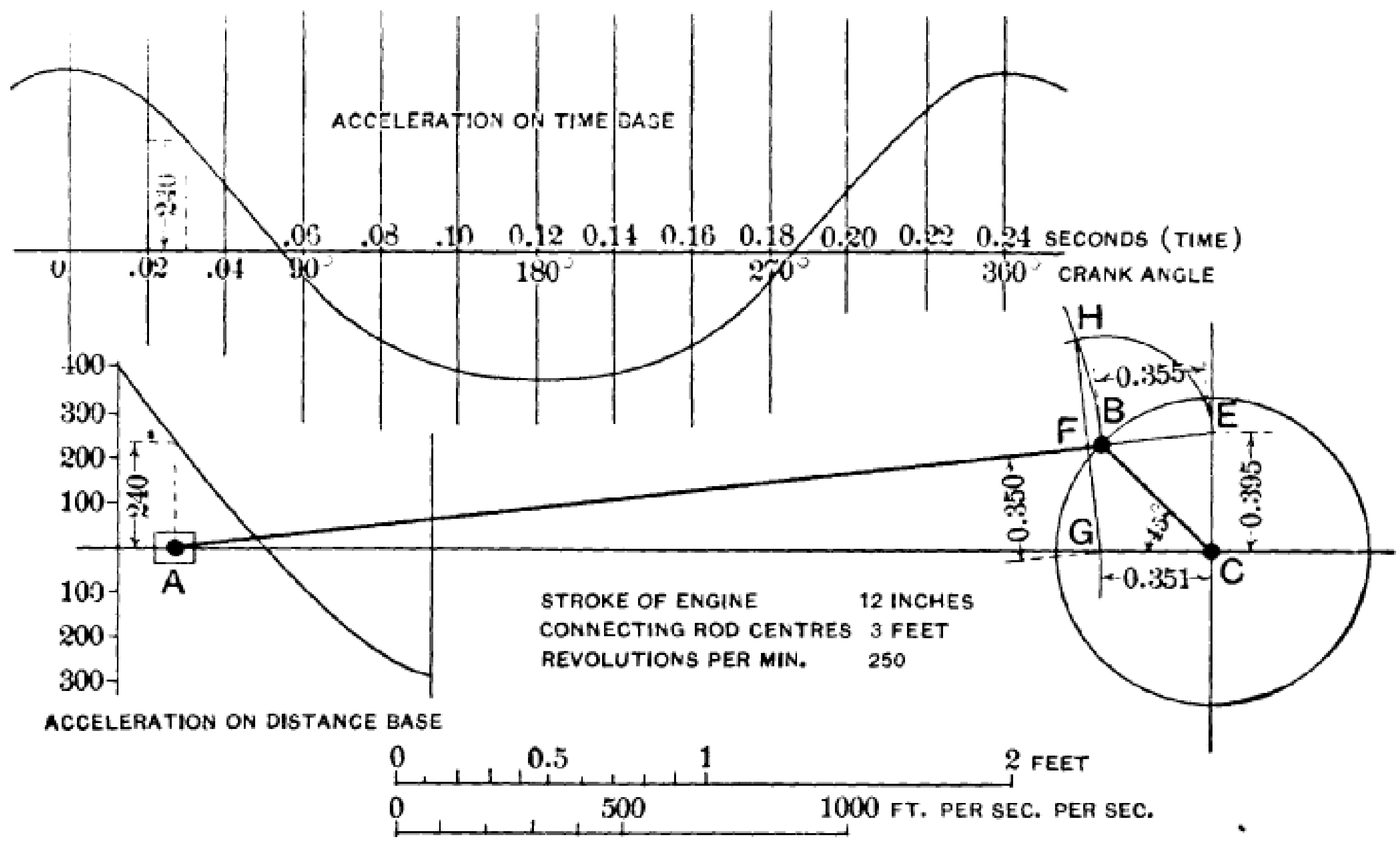

FIG. 67.

sents a velocity of 13.08 feet per second. Hence the velocity scale is 1 foot = 26.16 feet per second, or $\frac{V_c}{CB} = \frac{13.08}{0.5} = 26.16$.

It has been shown that the rate at which the distance CE is changing is

$$u_0 = V_c \cdot \frac{CG}{CB}$$

$$= 0.351 \times 26.16 = 9.18 \text{ feet per second.}$$

This, however, is not the numerical value of the acceleration required, for it represents the rate of change of a length CE, each foot of which stands for a velocity of 26.16 feet per second. Therefore, expressing u_0 in feet per second per second, we have

$u_0 = 0.351 \times 26.16 \times 26.16 = 240$ feet per second per second, a result agreeing (within the limits of accuracy for a small scale drawing) with that just calculated.

It is thus seen that to determine the scale to which CG

represents the piston acceleration, we find, first, the piston velocity represented by unit length of CE (in this case 26.16 feet per second); then it follows that a change of length of CE at the rate of one unit per second represents a change of piston velocity at the rate of 26.16 units per second, or a piston acceleration of 26.16 units. But each unit of length of CG has been shown to represent a change of length of CE at the rate of 26.16 units per second, so that, finally, unit length of CG represents a piston acceleration of 26.16×26.16 units.

This relation may be expressed by saying that if the engine were drawn out full size and the linear velocity scale were 1 foot $= n$ feet per second, then the acceleration scale would be 1 foot $= n^2$ feet per second per second. In this case, as in the case of all graphic methods of determining velocities and accelerations, the manner of finding the velocity and acceleration scales must be thoroughly understood; if this is not done, the diagram becomes almost useless, since no numerical values can be obtained from it.

A number of other constructions for the piston acceleration in the direct-acting engine have been devised.*

37. Angular Velocity and Acceleration of Connecting-rod.—To study the movement of the connecting-rod, adopting the same notation as in § 35, we have, as before,

$$\sin \varphi = \frac{\sin \theta}{n},$$

$$\cos \varphi = \frac{\sqrt{n^2 - \sin^2 \theta}}{n}.$$

The angular velocity of the connecting-rod is the rate of change of φ with regard to time, and we obtain at once

$$\frac{d\varphi}{dt} = \frac{\cos \theta}{\sqrt{n^2 - \sin^2 \theta}} \frac{d\theta}{dt}.$$

* See a paper by Prof. Elliott, Engineering, Vol. LIX, pp. 587 and 711, and Zeitschrift des V. Deutscher Ingenieure, Oct. 13, 1894.

Since $\frac{d\theta}{dt}$ is the angular velocity of the crank, we have

$$\text{angular velocity of connecting-rod} = \frac{V_c}{r}\frac{\cos\theta}{\sqrt{n^2-\sin^2\theta}}.$$

Differentiating again to find the angular acceleration, we obtain

$$\frac{d^2\varphi}{dt^2} = \frac{V_c}{r}\cdot\frac{d}{d\theta}\,\frac{\cos\theta}{\sqrt{n^2-\sin^2\theta}}\cdot\frac{d\theta}{dt}$$

$$= \frac{V_c^{\,2}}{r^2}\left\{\cos\theta\frac{d}{d\theta}(n^2-\sin^2\theta)^{-\frac{1}{2}} - \sin\theta(n^2-\sin^2\theta)^{-\frac{1}{2}}\right\}$$

$$= -\frac{V_c^{\,2}}{r^2}\cdot\frac{\sin\theta}{\sqrt{n^2-\sin^2\theta}}\cdot\frac{n^2-1}{n^2-\sin^2\theta}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For ordinary values of n it is sufficiently accurate to write approximately

$$\text{angular acceleration} = -\frac{V_c^{\,2}}{r^2}\cdot\frac{\sin\theta}{\sqrt{n^2-\sin^2\theta}}. \quad . \quad (2a)$$

Taking the same example as before, at a crank angle of 45° we have

$$\sin\theta = 0.70711, \quad n = 6, \quad V_c = 13.08, \quad r = 0.5.$$

Thus $\sqrt{n^2-\sin^2\theta} = 5.96$ and $\cos\theta = 0.70711$. Therefore

$$\text{angular velocity} = \frac{13.08 \times 0.70711}{0.5 \times 5.96} = 3.11 \text{ radians per second,}$$

and, from equation (2),

$$\text{angular acceleration} = -\frac{13.08^2}{0.25}\cdot\frac{0.70711}{5.96}\cdot 0.987$$

$$= -80.2 \text{ radians per second per second.}$$

Using equation (2a), we should obtain -81.2 as a result.

The simple construction of Figs. 65, 66, and 67 gives us the angular velocity of the connecting-rod. For

$$\text{angular velocity of crank} = \frac{V_c}{BC},$$

and $$\text{angular velocity of connecting-rod} = \frac{V_c}{BD}.$$