CHAPTER II.

POSITION, VELOCITY, AND ACCELERATION.

11. Velocity.—While Kinematics in its general sense comprises all kinds of problems dealing with pure motion, the number of such problems falling within the province of the Kinematics of Machines is somewhat limited. We shall consider in this chapter some elementary notions concerning velocity which are applicable to the purposes of the Kinematics of Machines. Methods of studying the position and motion of a point or rigid body from a geometrical point of view have already been indicated; it now remains to investigate not only the amount by which such position is changed during motion, but the rate of such change of position. Going a step farther still, it may be asked, does such velocity increase, diminish, or change in any way as time goes on, and if so, at what rate?

The rate of change of position of a point or body is called its *velocity*. A body, as we have seen, may change its position by a motion of translation, or by one of rotation. Hence we distinguish between linear and angular velocity. The former is measured by the space passed over in unit of time, and is usually expressed in feet per second, although other units, such as miles per hour or knots, are adopted in special cases. The latter is measured by the angle described in unit of time, the natural unit being therefore one radian per second. Engineers, however, commonly measure angular velocity in revolutions per minute. Either kind of velocity may be uniform or variable.

It is important to note that the term velocity involves the ideas of both speed, direction, and sense. In other words, a velocity is a vector quantity, and, like other vector quantities, may be represented by a straight line of definite length, this length being proportional to the speed, or magnitude of the velocity, measured in feet per second, radians per second, or whatever units are to be employed.

In the case of linear velocity the direction of the vector or straight line representing the velocity on the diagram is taken to represent the *direction* of the motion. Thus, for example, we might draw upon a map a line running east and west, and 2 inches in length, and take this line as representing a linear velocity of 2 miles per hour, or 2 feet per second, either from east to west, or from west to east. The *sense* of the motion may be either from east to west, or from west to east. In order to indicate the sense, we place upon the line a small arrow-head so as to show the point towards which the body is moving (see Fig. 12).

In the case of angular velocity the direction of the vector on the diagram would be taken to represent the direction in space of the axis about which the spin or rotation is taking place, and a line similar to that mentioned above would mean a spin of two radians per second, or two revolutions per minute, according to the scale, about an axis lying east and west. This rotation may be either right-handed or left-handed, and it is therefore customary to indicate the sense by placing the arrow-head in such a fashion that the spin will appear to be right-handed, or clockwise, when looking along the axis and following the arrow-head.

It is plain that in this manner a velocity, whether linear or angular, may be completely represented by a vector, having magnitude, direction, and sense.

12. Uniform Velocity.—A body having uniform velocity (whether angular or linear) performs equal changes of position in equal times. If the body has a uniform linear velocity v, it describes a distance vt in time t, where t is any number of units of time. Calling s the space described, we have therefore $s=vt$. Similarly, if the uniform velocity is angular and is denoted by ω, any line on the body in a plane perpendicular to the axis of rotation describes ω radians in each second and therefore ωt radians in t seconds. Hence, calling θ the angle described in t seconds, we have

$$\theta=\omega t.$$

If a point, at distance r from the centre about which it moves in a circular path, has a linear velocity v, its angular velocity is measured by the angle subtended at the centre by the path it describes in one second. Hence

$$\omega=\frac{v}{r} \quad \text{or} \quad v=\omega r.$$

13. **Variable Velocity.** — In general a moving body varies its speed as well as its direction of motion. It is easy by observing the time taken to travel over a known distance, for example in a train, to calculate the *average speed* of the train during the interval considered. This does not tell us, however, the *actual speed of the train at any instant* during the interval of time, which may be quite different from the average speed.

The velocity at any instant, or instantaneous velocity, is measured by the space (or angle, as the case may be) which would have been described in a unit of time if the motion had continued uniformly, during that interval, at the same rate as at the instant considered. The word instant is here used to mean an indefinitely small interval of time.

We are not able to measure the distance (or angle) described during an indefinitely small interval of time, and therefore have to obtain the value of the instantaneous velocity of a body in another manner.

This will be best understood by a numerical example. Suppose that a man in a street-car at 12 o'clock finds that in 10 seconds the car traverses a distance of 200 feet. This gives 20 feet per second as the average speed during the 10 seconds after 12 o'clock. Suppose that other observations taken during the first $1\frac{1}{5}$, 2, and 4 seconds showed that during these times the car travelled 30, 48, and 100 feet, corresponding to average speeds of 25, 24, and 21.75 feet per second. It is evident that the speed must really have been continually diminishing, and that the shorter the time during which the observation was made, the more nearly do we

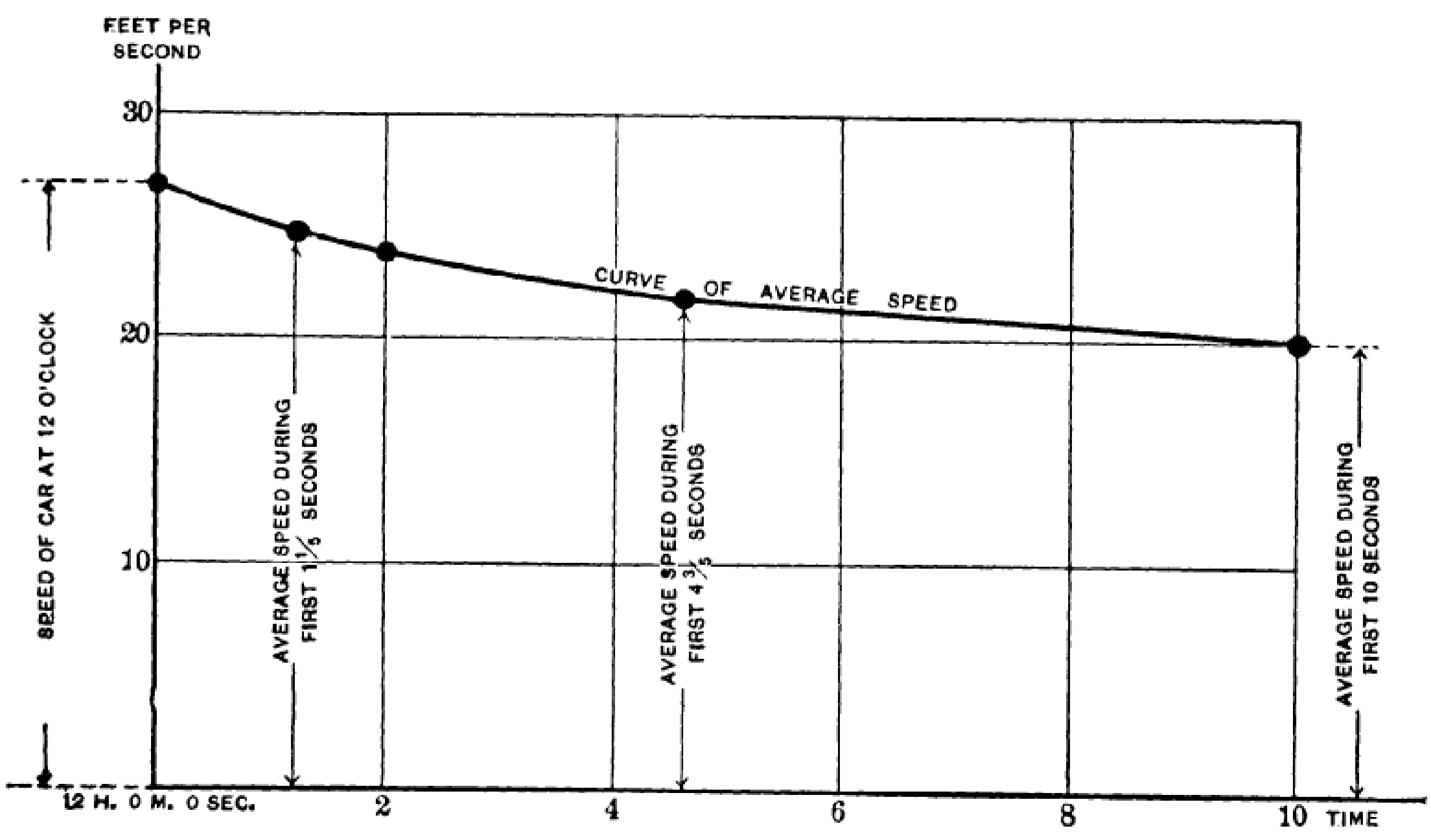

FIG. 10.

obtain the speed at which the car must have been travelling at 12 o'clock. To arrive at this more exactly, since we cannot measure the distance passed over in an infinitely small interval of time, we plot a curve from our observations, as in Fig. 10, and see that the speed at 12 o'clock must have been 27 feet per second. In mathematical language, if Δs be the distance traversed in a small interval of time Δt, the average velocity during that small interval is $\frac{\Delta s}{\Delta t}$, while the velocity at the instant beginning the interval is measured by diminishing Δt indefinitely, and finding the limiting value of $\frac{\Delta s}{\Delta t}$, or, in the language of the calculus $\frac{ds}{dt}$. Thus

$$v=\frac{ds}{dt}.$$

The same reasoning, of course, applies in the case of angular velocity, where we should write

$$\omega=\frac{d\theta}{dt}.$$

Compare these with the corresponding expressions in the case of uniform velocity.

14. **Uniform Acceleration.**—A body moving with uniform acceleration changes its velocity by equal amounts in equal times. Thus suppose that in time t the velocity changes from v_1 to v_2; we have, if a is the acceleration,

$$a=\frac{v_2-v_1}{t}. \qquad (1)$$

Again, the average velocity during the time t is $\frac{v_2+v_1}{2}$, the arithmetical mean between the initial and final velocities; hence if s be the space described,

$$s=\frac{v_2+v_1}{2}t. \qquad (2)$$

From these two expressions we find

$$s=\frac{v_2^2-v_1^2}{2a}. \qquad (3)$$

But $v_2 = v_1 + at$. Substituting in (3), we get

$$s = v_1 t + \frac{a}{2} t^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In the case of angular velocity precisely similar relations hold, so that, calling α the uniform angular acceleration, ω the angular velocity at the beginning of the time t, and θ the angle described, we have, instead of (4),

$$\theta = \omega_1 t + \frac{\alpha}{2} t^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4a)$$

To express the velocity in terms of distance (or angle) and initial velocity we shall have instead of (3)

$$\omega_2^2 = \omega_1^2 + 2\alpha\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3a)$$

while the expression connecting velocity, acceleration, and time is

$$\omega_2 = \omega_1 + \alpha t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

As an example of the use of these expressions, suppose a wheel is revolving thirty times per second and comes to rest in 12 seconds. How many revolutions will it make in coming to rest if uniformly retarded?

We have $\omega_2 = \omega_1 + \alpha t$: hence

$$12\alpha + 30 \times 2\pi = 0,$$

and $\alpha = -\frac{60\pi}{12} = -15.71$ radians per second per second.

Again, $\omega_2^2 = \omega_1^2 + 2\alpha\theta$; hence

$$(60\pi)^2 - 10\pi\theta = 0,$$

and $\theta = \frac{(60\pi)^2}{10\pi} = 360\pi$ radians.

Hence the wheel comes to rest in 180 revolutions.

Again, a train starting from rest has a uniform acceleration of half a mile per hour per second. How far will it have travelled before attaining a speed of 30 miles per hour, and in what time will this occur?

In the equation (4) above we have $s = v_1 t + \frac{1}{2}at^2$.

Here $v_1 = 0$, t evidently will be 60 seconds, and $a = \frac{2640}{3600} =$ 0.733 feet per second per second. Thus

$$s = \frac{0.733 \times 3600}{2} = 1218 \text{ feet.}$$

It should be noted that it is as incorrect to speak of an acceleration of so many *feet per second* as it would be to say that a body has a velocity of so many *feet*, without mentioning the unit of time.

15. Acceleration in General. — The determination of velocity and acceleration in the case of non-uniform or non-

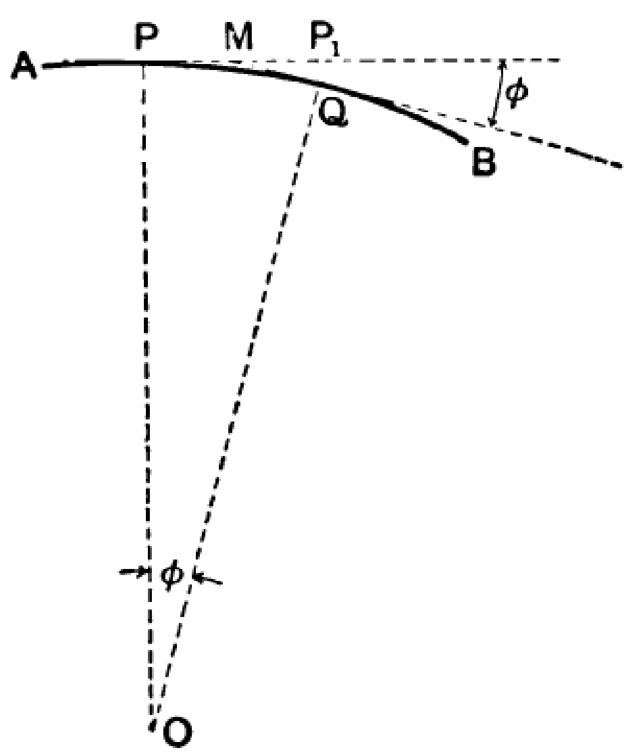

FIG. 11.

uniformly accelerated motion will be discussed later. Acceleration is defined generally as rate of change of velocity with regard to time, and so far we have used the term as meaning change in the *magnitude* of the velocity, whether linear or angular. Strictly speaking, however, a change in the *direction* of motion in linear velocity, or in the position of the axis of rotation in angular velocity, is also an acceleration. In fact, a point travelling in a circular path around a fixed point has an acceleration impressed upon it, although its angular velocity may be uniform, and such acceleration is called *radial*, for reasons which will presently be seen.

In Fig. 11 let AB represent a portion of the curved path alng which a point is travelling with a linear velocity v, whose direction is continually changing. Let ρ be the radius of curvature OP of a very small portion PQ, and O the centre of curvature, φ being the very small angle between the tangents at P and Q, an angle so small that the arc PQ is not sensibly different from its chord.

Consider the acceleration in a direction parallel to PO. The time taken for the particle to travel from P to Q will be $\frac{PQ}{v}$. But during this time the distance traversed under acceleration a is P_1Q parallel to PO. Hence (if a is constant)

$$P_1Q = \tfrac{1}{2}a\left(\frac{PQ}{v}\right)^2 \text{ and } a = v^2\frac{2P_1Q}{(PQ)^2}.$$

It is known that for very small angles the numerical value of the sine of an angle is sensibly the same as the angle itself (of course expressed in circular measure). Also in the figure if we make φ small enough, $MQ = \tfrac{1}{2}PQ$, the error in this statement diminishing as φ diminishes. Hence *for an indefinitely small value of* φ we may say that

$$\frac{2P_1Q}{PQ^2} = \frac{P_1Q}{PQ.MQ} = \phi \times \frac{1}{PQ} = \frac{PQ}{OP} \times \frac{1}{PQ} = \frac{1}{\rho} \text{ and hence } a = \frac{v^2}{\rho}.$$

The statement is exactly correct, and a is the radial acceleration at P, because we have taken φ as being the angle described during an indefinitely small interval of time.

The earth's equatorial radius is 4000 miles, and the earth makes one rotation on its axis in about 86,200 seconds. What is the radial acceleration of a particle on the earth's surface at the equator?

Linear velocity of a point at equator

$$=\frac{2\pi\times4000\times5280}{86200}\text{ feet per second.}$$

Thus $$\frac{v^2}{\rho}=\frac{(2\pi)^2\times4000\times5280}{86200^2}$$

$$=0.112\text{ feet per second per second.}$$

It is often necessary to find the acceleration of a body along its virtual radius; this is of course determined in exactly the same way as the radial acceleration with regard to a permanent centre.

16. Composition of Velocities and Accelerations.—It has been already pointed out that velocities, whether linear or angular, can be represented by straight lines of definite length, sense, and direction, and are in fact *vector quantities*, as distinguished from *scalar quantities*, such as mass, energy, and so on which have simply numerical values. Accelerations are also vector quantities.

The *resolved part* of a vector in any new direction is found by projecting its original length on the new direction.

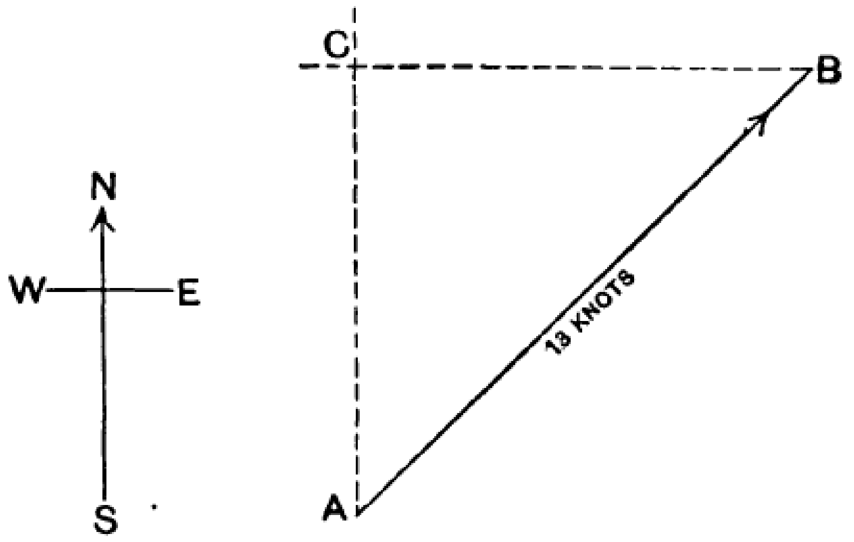

FIG. 12.

If, for example, a ship is proceeding northeast at a speed of 13 knots, represented by the vector AB (1 knot being a speed of 6080 feet, or 1 nautical mile, per hour), its resolved velocity in a northerly direction is represented by

$$AC = AB \cos 45^\circ = 13 \times 0.707 = 9.19 \text{ knots.}$$

This shows that each hour the position of the ship is 9.19 nautical miles farther to the northward.

Again, suppose that the ship, still steering N.E. at the same speed, runs into a current whose speed is 4 knots due east, what will be the real velocity of the ship relatively to the earth? *Relatively to the water* its speed is still 13 knots

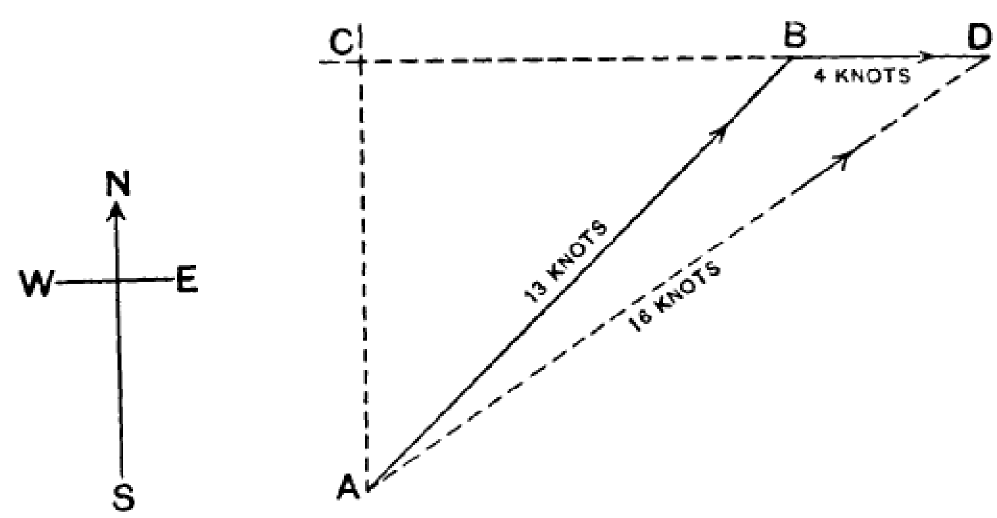

FIG. 13.

in a N.E. direction, but the water is itself moving, and at the end of the hour the ship will evidently be at D, a position obtained by measuring 4 nautical miles east from B. On calculation it will be found that at any time during the hour the ship has been moving relatively to the earth along the line AD, and its real speed over the ground (about 16 knots) will be measured by the length of AD, the third side of a triangle, whose other two sides represent respectively the velocity of the ship relatively to the water, and the velocity of the water relatively to the earth. We say, then, that the vector AD represents the *resultant* of the two vectors AB and BD, obtained by the process of vector addition.

The above example deals with plane motion in a straight line only. But if we are treating of the motion in space of a body having six degrees of freedom, its motion may be considered as made up of three motions of simple translation and three motions of rotation, which, when compounded according to the method just explained, constitute the actual motion of the body.

It must not be forgotten that the resultant of two or more angular velocities can be found in exactly the same way as for linear velocities. As already explained, it is customary to indicate an angular velocity by a vector (as in Fig. 14), representing the numerical value of the velocity by the length AB, the direction of the axis by the direction of AB, and the sense of rotation by drawing AB in such a manner that the rotation is clockwise, or right-handed, when looking from A to B. It is often necessary to compound or to resolve spins or angular velocities according to the method of vector addition, which will be already familiar to most readers under the name of the triangle of velocities, or the parallelogram law for the composition of vectors.

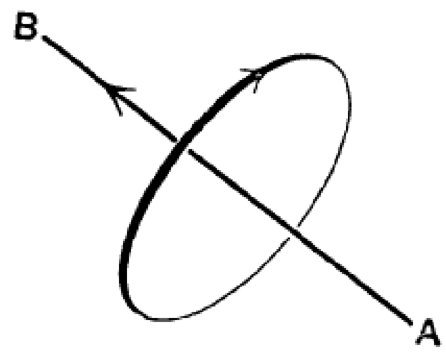

FIG. 14.

17. Resultant Acceleration.—In Fig. 15 let AB represent the original velocity of a particle, and suppose that accelerations represented by BC, BD are impressed upon the particle. Then BC and BD may be taken to represent the velocities generated in one second, corresponding respectively to the two accelerations.

If now the acceleration BC had alone acted on the point, its velocity at the end of one second would have been AC. Again, if AC had been the original velocity and an acceleration BD had been impressed, the final velocity at the end of one second would have been AD', where CD' is equal and parallel to BD. The two accelerations, therefore, have changed the original velocity from AB to AD'. But this effect would have been produced by compounding with AB *for one second* a velocity BD', and we may therefore look on BD' as representing the change of velocity in one second, due to the action of the accelerations BC and BD. In

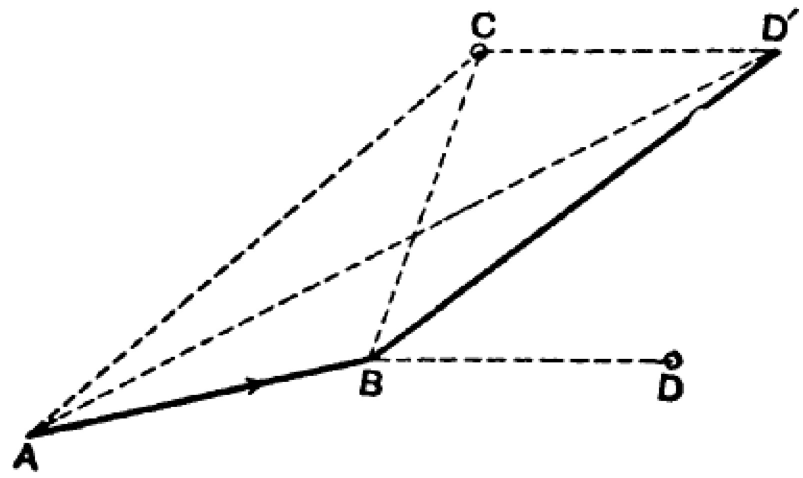

FIG. 15.

other words, BD' is the resultant of the accelerations BC and BD.

The general rule for the composition or addition of vectors, then, is that the resultant of two vectors is the diagonal of a parallelogram (or the third side of a triangle) of which the two components form the two adjacent sides. In this way we can find the resultant of any number of velocities, or of accelerations, either linear or angular. The same rules apply to the composition of any other vector quantities.

18. Diagrams of Displacement and Velocity.—In studying the motion of a body, whether linear or angular, it is necessary to know the position of the body at every instant during the motion, if we desire full information as to its velocity and acceleration. We have seen that if we only know the position of the body at certain times we can obtain the value of the average velocity between those times, but cannot tell exactly how the real velocity has changed.

It would of course be very cumbersome to have to state in words or figures a sufficient number of particulars to give us a practically complete knowledge of the position, velocity, and acceleration of a body, and therefore in such cases graphic methods of representation and calculation are generally adopted.

For example, in order to determine the velocity of a body whose changes of position are known, such a diagram as Fig. 16 is constructed. Two axes, *OA*, *OB*, are drawn at right angles, and distances measured parallel to *OA* according to any convenient scale are considered to represent time,

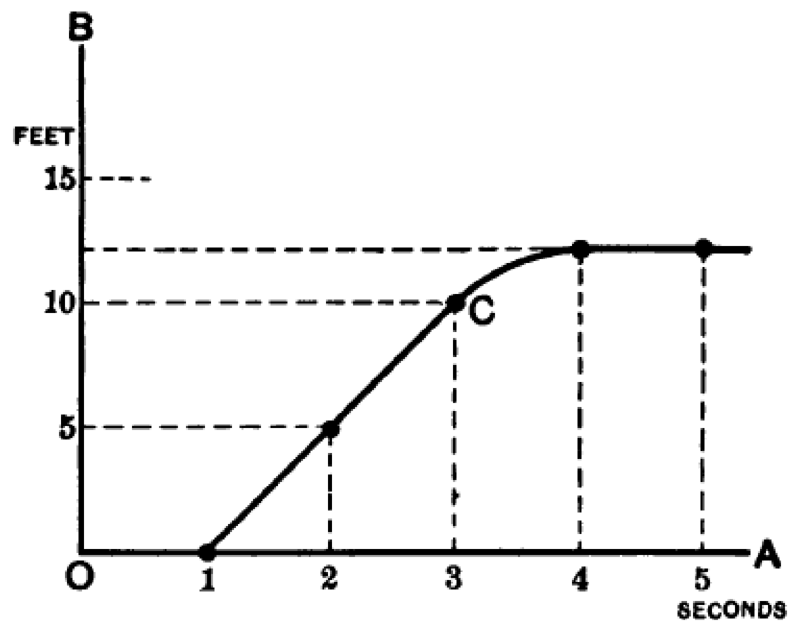

FIG. 16.

while lengths measured parallel to *OB* represent either the distance that has been traversed by the body, reckoning from some known position, or the angle turned through by the body at any given instant. This quantity we may call the displacement of the body, and it may be either linear or angular.

For instance, from the figure we see that after the lapse of 3 seconds the body in question has moved 10 feet from its original position, and we might give the information contained in the diagram in a less complete form in the shape of a table, thus:

Time........	0	1	2	3	4	5 seconds.
Displacement.	0	0	5	10	12	12 feet.

From the diagram, however, we are enabled to gather further particulars, for it is plain that the curve of displace-